Problem 1 (10 points)

Let $u_1$ and $u_2$ be vectors such that $||u_1|| = ||u_2|| = 1$, and $\langle u_1, u_2 \rangle = 0$. For any vector $x$, we define $P(x)$ as the vector $P(x) = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2$.

1. How would you geometrically interpret $P(x)$? (Hint: Think about projections)
2. Show that: $||P(x)||^2 = \langle x, u_1 \rangle^2 + \langle x, u_2 \rangle^2$.
3. Using parts (1) and (2), show that $||P(x)|| \leq ||x||$. When is $||P(x)|| = ||x||$?

Problem 2 (10 points)

Given two column vectors $x$ and $y$ in $d$-dimensional space, the outer product of $x$ and $y$ is defined to be the $d \times d$ matrix $x \circ y = xy^\top$.

1. Show that for any $x$ and $y$, $x^\top (x \circ y)y = ||x||^2 ||y||^2$. When is this equal to $x^\top \langle x, y \rangle y$?
2. Show that for any non-zero $x$ and $y$, the outer product $x \circ y$ always has rank 1.
3. Let $x_1, \ldots, x_n$ be $n \times 1$ data vectors, and let $X$ be the $n \times d$ data matrix whose $i$-th row is the row vector $x_i^\top$. Show that:

   $$X^\top X = \sum_{i=1}^n x_i \circ x_i$$

Problem 3 (10 points)

Suppose $A$ and $B$ are $d \times d$ matrices which are symmetric (in the sense that $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ for all $i$ and $j$) and positive semi-definite. Also suppose that $u$ is a $d \times 1$ vector such that $||u|| = 1$. Which of the following matrices are always positive semi-definite, no matter what $A$, $B$ and $u$ are? Justify your answer.

1. $10A$.
2. $A + B$.
3. $uu^\top$.
5. $I - uu^\top$ (Hint: Write down $x^\top (I - uu^\top)x$ in terms of some dot-products, and try using Cauchy-Schwartz.)
Problem 4 (10 points)

In class, we discussed how to define a norm or a length for a vector. It turns out that one can also define a norm or a length for a matrix. Two popular matrix norms are the Frobenius norm and the spectral norm. The Frobenius norm of a $m \times n$ matrix $A$, denoted by $\|A\|_F$ is defined as:

$$
\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}
$$

The spectral norm of a $m \times n$ matrix $A$, denoted by $\|A\|$ is defined as:

$$
\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}
$$

where $x$ is a $n \times 1$ vector.

1. Let $I$ be the $n \times n$ identity matrix. What is its Frobenius norm? What is its spectral norm? Justify your answer.

2. Suppose $A = uv^\top$ where $u$ is a $m \times 1$ vector and $v$ is a $n \times 1$ vector. Write down the Frobenius norm of $A$ as function of $\|u\|$ and $\|v\|$. Justify your answer.

3. Write down the spectral norm of $A$ in terms of $\|u\|$ and $\|v\|$. Justify your answer.