Today's learning goals

- Apply the Pumping Lemma in proofs of nonregularity
- Identify some nonregular sets

**Exam 1** Wednesday April 25, in class
- Request special seats via Google form by end of class today!
- Seat assignments posted on Piazza tomorrow
- Review: in-class on Monday and Monday evening 7-9pm  GH 242
Pumping Lemma

If \( A \) is a regular language, then there is a number \( p \) (the pumping length) where, if \( s \) is any string in \( A \) of length at least \( p \), then \( s \) may be divided into three pieces, \( s = x y z \) such that:

\[
\begin{align*}
\textbullet & \quad |y| > 0, \text{ and} \\
\textbullet & \quad \text{for each } i \geq 0, \ xy^i z \in A,
\end{align*}
\]

\( s \) can be cut into 3 pieces to give us lots of new strings.
Pumping Lemma

If $A$ is recognized by DFA $M$ with state diagram below, the computation of $M$ on any string $s$ of length $\geq p = |Q|$ must have a loop. Divide $s$ into the strings labelling the path before the loop $x$, the loop itself $y$, and from the loop to the accept state $z$.\[x y z \in L(M) = A\]
\[x y^2 z \in L(M)\]
\[x z \in L(M)\]
\[x y^n y z \in L(M)\]
Pumping Lemma

- True for all (but not only) regular sets.
- Can't be used to prove that a set is regular
- Can be used to prove that a set is not regular … how?

Really? Finite sets?
Negation

flash-back to CSE 20 😊

• **Pumping lemma**  "There is \( p \), where \( p \) is a pumping length for \( L \)"

• Given a specific number \( p \), it being a pumping length for \( L \) means

\[
\forall s \left( |s| \geq p \land s \in L \right) \rightarrow \exists x \exists y \exists z \left( s = xyz \land |y| > 0 \land |xy| \leq p \land \forall i \left( xy^i z \in L \right) \right)
\]

• So \( p \) not being a pumping length of \( L \) means

\[
\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( \left( s = xyz \land |y| > 0 \land |xy| \leq p \right) \rightarrow \exists i \left( xy^i z \notin L \right) \right) \right)
\]
Proof strategy
To prove that a language L is not regular

• Consider arbitrary positive integer p.
• Prove that p isn't a pumping length for L.

• Conclude that L does not have any pumping length and is therefore not regular.
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ \textit{(the pumping length)} where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = x y z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $x y^i z \in A$,
- $|x y| \leq p$.

How does this apply to some known regular languages, e.g. the set of all strings, or \{a, ab\}
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider any positive integer \( p \). We wish to show that \( p \) is not pumping length for \( L \).

i.e. counterexample

\[ S = 0^{p}1^{p} \]

Let \( S \in L \) with \( |S| > p \). i.e. cannot be pumped.
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof: Consider an arbitrary positive integer. WTS $p$ is not a pumping length for $L$.

*How?* Want to show that there is some string that *should* be pump'able but isn't.

Conclude that $L$ does not have any pumping length and is therefore not regular.
Using the Pumping Lemma

L = \{0^n1^n \mid n \geq 0\}  **CLAIM:** p is not a pumping length for L.

*How would you prove the claim?*

A. Find a string with length \( \geq p \) that is not in L.

B. Find a string with length \(< p\) that is in L.

C. None of the above.

\[ \exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right) \]
Using the Pumping Lemma

$L = \{0^n1^n \mid n \geq 0\}$ **CLAIM:** $p$ is not a pumping length for $L$.

**WTS**

$$\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right)$$

Find a string $s$ such that

1. $|s| \geq p$
2. $s$ is in $L$
3. No matter how we cut $s$ into three (viable) pieces, some related string obtained by repeating the middle part falls out of $L$. 
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).
Consider the string

\[ s = 0^p1^p. \]

1. \(|s| \geq p \)? \( \text{yes} : \ |s| = 2p \)
2. \( s \) is in \( L \)? \( \text{yes} \)
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by repeating the middle part falls out of \( L \)?
Using the Pumping Lemma

**Claim:** The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

**Proof:** Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \). Consider the string \( s = 0^p1^p \). Then, \( s \) is in \( L \) and \( |s| = 2p \geq p \). Consider any division of \( s \) into three parts 

\[ s = xyz \text{ with } |y| > 0, \ |xy| \leq p. \]

Since \( |xy| \leq p \), \( x = 0^k \), \( y = 0^m \), \( z = 0^r1^p \) with \( k+m+r = p \), and since \( |y| > 0 \), \( m > 0 \). Picking \( i = 0 \): \( xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p \), which is not in \( L \) because \( k+r < p \). Thus, no \( p \) can be a pumping length for \( L \) and \( L \) is not regular.
Proof strategy

To prove that a language \( L \) is \textbf{not} regular

- Consider arbitrary positive integer \( p \).
- Prove that \( p \) isn't a pumping length for \( L \).

- Conclude that \( L \) does not have any pumping length and is therefore \textbf{not} regular.
Picking s

To complete proofs with Pumping Lemma, we will need to build (useful) examples of strings with length $\geq p$ that are in a given language.

- $L_1 = \{ a^m b^m a^n | m, n \geq 0 \}$
- $L_2 = \{ w w | w \text{ is a string over } \{0,1\} \}$
- $L_3 = \{ w w^R | w \text{ is a string over } \{0,1\} \}$

$S_1 = a^{p^2} b^{p^2} a^{p^2} \quad a^{p^2} b^{p^2} a^{p^2}$

$S_2 = 0^{p^2} 0^{p^2} \quad 0^{p^2} \cdot 1^{p^2}$

$S_3 = 0^{p^2} 1^{p^2} 0^{p^2} \quad 0^{p^2} 1^{p^2}$
Another example

Claim: The set \( \{ w w^R \mid w \text{ is a string over \{0,1\}} \} \) is not regular.

Proof: ... You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \) and \( s \) "can't be pumped" ... Consider \( i = \ldots \)

Which \( s \) and \( i \) let us complete the proof?

A. \( s = 0^p0^p, i = 2 \)  
B. \( s = 0110, i = 0 \)  
C. \( s = 0^p110^p, i = 1 \)  
D. \( s = 1^p001^p, i = 3 \)  
E. I don't know

\( |s| \geq p \)
How do we choose i?

Claim: The set \( \{0^j1^k \mid j,k \geq 0 \text{ and } j \geq k \} \) is not regular.

Proof: … You must pick s carefully: we want \(|s|\geq p\) and s in L and s "can't be pumped" … Consider i=…

Which s and i let us complete the proof?
A. s = 0^p1^p, i=2  B. s = 0^p1^p, i=p  C. s = 0^p1^p, i=1  D. s = 0^p1^p, i=0  E. I don't know
Another example

Claim: The set \( \{a^mb^ma^n \mid m,n \geq 0\} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \) and \( s\) "can't be pumped"

Which choices of \( s \) could we have used in the proof?
A. \( s = a^pb^p \)  
B. \( s = aba \)  
C. \( s = a^pb^pa^p \)  
D. \( s = b^p \)  
E. None of the above
Do we always need Pumping Lemma?

Claim: The set

\{w \mid w \text{ has different } \#s \text{ of } 0\text{s and } 1\text{s OR has a } 1 \text{ before a } 0\}\n
is nonregular.

Proof:
Regular sets: not the end of the story

- Many **nice / simple / important** sets are not regular
- Limitation of the finite-state automaton model
  - Can't "count"
  - Can only remember finitely far into the past
  - Can't backtrack
  - Must make decisions in "real-time"
- We know computers are more powerful than this model…

Which conditions should we relax?
For next time

• Work on Practice questions for Exam 1

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