Today's learning goals

• Apply the Pumping Lemma in proofs of nonregularity
• Identify some nonregular sets
Pumping Lemma

If $A$ is a regular language, then there is a number $p \ (\text{the pumping length})$ where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = x \ y \ z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

How does this apply to some known regular languages, e.g. the set of all strings, or $\{a, \ ab\}$
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).

How? Want to show that there is some string that *should* be pump'able but isn't.

Conclude that \( L \) does not have any pumping length and is therefore not regular.
Using the Pumping Lemma

L = \{0^n1^n \mid n \geq 0\}  \textbf{CLAIM:} p is not a pumping length for L.

How would you prove the claim?

A. Find a string with length \( \geq p \) that is not in L.
B. Find a string with length \(<p\) that is in L.
C. None of the above.

$$\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right)$$
Using the Pumping Lemma

L = \{0^n1^n \mid n \geq 0\}  \textbf{CLAIM:}  p \text{ is not a pumping length for } L.

\textbf{WTS}

\exists s (|s| \geq p \land s \in L \land \forall x \forall y \forall z ((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L)))

Find a string s such that
1. \hspace{1em} |s| \geq p
2. \hspace{1em} s \text{ is in } L
3. \hspace{1em} \text{No matter how we cut } s \text{ into three (viable) pieces, some related string obtained by repeating the middle part falls out of } L.
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.
Proof: Consider an arbitrary positive integer. WTS $p$ is not a pumping length for $L$. Consider the string

$s = 0^p1^p$.

1. $|s| \geq p$ ?
2. $s$ is in $L$ ?
3. No matter how we cut $s$ into three (viable) pieces, some related string obtained by repeating the middle part falls out of $L$ ?
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \). Consider the string \( s = 0^p1^p \). Then, \( s \) is in \( L \) and \(|s| = 2p \geq p\). Consider any division of \( s \) into three parts \( s = xyz \) with \(|y|>0, |xy|\leq p\).

Since \(|xy|\leq p\), \( x=0^k\), \( y=0^m\), \( z=0^r1^p\) with \( k+m+r=p\), and since \(|y|>0, m>0\). Picking \( i=0\): \( xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p\), which is not in \( L \) because \( k+r < p\). Thus, no \( p \) can be a pumping length for \( L \) and \( L \) is not regular.
Proof strategy

To prove that a language $L$ is not regular

- Consider arbitrary positive integer $p$.
- Prove that $p$ isn't a pumping length for $L$.

- Conclude that $L$ does not have any pumping length and is therefore not regular.
Another example

Claim: The set \( \{a^m b^m a^n \mid m,n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).
Consider the string

\[ s = ??? \]

1. \(|s| \geq p \) ?
2. \( s \) is in \( L \) ?
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by "repeating" the middle part falls out of \( L \) ?
Aside…

To complete proofs with Pumping Lemma, we will need to build (useful) examples of strings with \( \text{length} \geq p \) that are \textit{in} a given language.

- \( L_1 = \{ a^n b^m a^n \mid m, n \geq 0 \} \)
- \( L_2 = \{ \text{ww} \mid \text{w is a string over \{0,1\}} \} \)
- \( L_3 = \{ \text{ww}^R \mid \text{w is a string over \{0,1\}} \} \)
Another example

Claim: The set \( \{a^nb^ma^n \mid m,n \geq 0\} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \( |s| \geq p \) and \( s \) in \( L \) and \( s \) "can't be pumped"

Which other choices of \( s \) could we have used in the proof?

A. \( s = a^pb^p \)  
B. \( s = aba \)  
C. \( s = a^pb^pa^p \)  
D. \( s = b^p \)  
E. None of the above
And another

**Claim:** The set \{w \ w^R \mid w \text{ is a string over } \{0,1\} \} is not regular.

**Proof:** ... You must pick \(s\) carefully: we want \(|s| \geq p\) and \(s\) in \(L\) and \(s\) "can't be pumped" ... Consider \(i=\ldots\)

Which \(s\) and \(i\) let us complete the proof?

A. \(s = 0^p0^p, i=2\)  
B. \(s = 0110, i=0\)  
C. \(s = 0^p110^p, i=1\)  
D. \(s = 1^p001^p, i=3\)  
E. I don't know
How do we choose i?

Claim: The set \( \{0^j1^k \mid j, k \geq 0 \text{ and } j \geq k \} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \) and \( s \) "can't be pumped" … Consider \( i = \ldots \)

Which \( s \) and \( i \) let us complete the proof?
A. \( s = 0^p1^p, \ i = 2 \)  B. \( s = 0^p1^p, \ i = p \)  C. \( s = 0^p1^p, \ i = 1 \)  D. \( s = 0^p1^p, \ i = 0 \)  E. I don't know
Do we always need Pumping Lemma?

**Claim**: The set

\{w \mid w \text{ has different } \# \text{ of 0s and 1s OR has a 1 before a 0}\}

is nonregular.

**Proof:**
Regular sets: not the end of the story

- Many nice / simple / important sets are not regular
- Limitation of the finite-state automaton model
  - Can't "count"
  - Can only remember finitely far into the past
  - Can't backtrack
  - Must make decisions in "real-time"
- We know computers are more powerful than this model…

*Which conditions should we relax?*
For next time

• Work on Practice questions for Exam 1

• **Exam 1** April 25, in class
  • Seat assignments on Piazza
  • Review: in-class on Monday and Monday evening 7-9pm  GH 242