Today's learning goals

- Explain the limits of the class of regular languages
- Justify why the Pumping Lemma is true
- Apply the Pumping Lemma in proofs of nonregularity
- Identify some nonregular sets
Proving nonregularity

How can we prove that a set is non-regular?

A. Try to design a DFA that recognizes it and, if the first few attempts don't work, conclude there is none that does.
B. Prove that it's a strict subset of some regular set.
C. Prove that it's the union of two regular sets.
D. Prove that its complement is not regular.
E. I don't know.
Counting languages

How many languages over \{0,1\} are there?

A. Finitely many because \{0,1\} is finite.
B. Finitely many because strings are finite.
C. Countably infinitely many because \{0,1\}\^* is countably infinite.
D. Uncountably many because languages are in the power set of \{0,1\}\^*.
E. None of the above.
Counting regular languages over \{0,1\}

\[ | \{ \text{regular languages} \} | \leq | \{ \text{regular expressions} \} | \]

Each regular expression is a finite string over the alphabet

\[ \{ 0, 1, \varepsilon, \varnothing, (, ), U, * \} \]

The set of strings over an alphabet is countably infinite.

Conclude: countably infinitely many regular languages.
Where we stand

Fact 1: There exist nonregular languages.

Fact 2: If we know some languages are nonregular, we can conclude others must be too. (cf. Discussion)

But, we don't have any specific examples of nonregular languages.

Yet.
Bounds on DFA

- in DFA, memory = states

- Automata can only "remember"…
  - …finitely far in the past
  - …finitely much information

- If a computation path visits the same state more than once, the machine can't tell the difference between the first time and future times it visited that state.
Example!

\{ 0^n1^n \mid n \geq 0 \}

*What are some strings in this set?*

*What are some strings not in this set?*

*Is this set finite or infinite?*

*Compare to \( L(0^*1^*) \)*

*Design a DFA? NFA?*
Example!

\{ 0^n1^n \mid n \geq 0 \}

What are some strings in this set?
What are some strings not in this set?
Is this set finite or infinite?

Compare to \( L(0^*1^*) \)
Design a DFA? NFA?
Pumping

- Focus on computation path through DFA
Pumping

- Focus on computation path through DFA
Pumping

- Focus on computation path through DFA

Idea: if one long string is accepted, then many other similar strings have to be accepted too.
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = x y z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^i z \in A$, 
- $|xy| \leq p$. 

Pumping Lemma

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- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$. 

Sipser p. 78 Theorem 1.70
Pumping Lemma

If $A$ is recognized by DFA $M$ with state diagram below, the computation of $M$ on any string $s$ of length $\geq p = |Q|$ must have a loop. Divide $s$ into the strings labelling the path before the loop $x$, the loop itself $y$, and from the loop to the accept state $z$. 
Pumping Lemma

If A is recognized by DFA M with state diagram below, the computation of M on any string s of length $\geq p = |Q|$ must have a loop. Divide s into the strings labelling the path before the loop $x$, the loop itself $y$, and from the loop to the accept state $z$.

Which of the following is true?

A. $|xy| \leq p$
B. $|y| > 0$
C. $xy^iz$ is accepted by M for all $i$
D. All of A,B,C
E. None of them
Pumping Lemma

• True for **all** (but not only) regular sets.

  • Can't be used to prove that a set **is** regular  Ex 1.49
  • Can be used to prove that a set **is not** regular … how?
Negation

- Pumping lemma ``There is \( p \), where \( p \) is a pumping length for \( L \)"

- Given a specific number \( p \), it being a pumping length for \( L \) means

\[
\forall s \left( |s| \geq p \land s \in L \right) \rightarrow \exists x \exists y \exists z \left( s = xyz \land |y| > 0 \land |xy| \leq p \land \forall i (xy^i z \in L) \right)
\]

- So \( p \) not being a pumping length of \( L \) means

\[
\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^i z \notin L) \right) \right)
\]
Proof strategy

To prove that a language $L$ is not regular

- Consider arbitrary positive integer $p$.
- Prove that $p$ isn't a pumping length for $L$.
- Conclude that $L$ does not have any pumping length and is therefore not regular.
For next time

• Work on Group Homework 2  
  due Saturday

Pre class-reading for Friday: Examples 1.75, 1.77.
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof: Consider an arbitrary positive integer. WTS $p$ is not a pumping length for $L$.

How? Want to show that there is some string that *should* be pump'able but isn't.
Using the Pumping Lemma

$L = \{0^n1^n | n \geq 0\}$ CLAIM: $p$ is not a pumping length for $L$.

How would you prove the claim?

A. Find a string with length $\geq p$ that is not in $L$.
B. Find a string with length $<p$ that is in $L$.
C. None of the above.

$\exists s \ (|s| \geq p \land s \in L \land \forall x \forall y \forall z \ ((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L)))$
Using the Pumping Lemma

$L = \{0^n1^n \mid n \geq 0\}$ **CLAIM:** $p$ is not a pumping length for $L$.

**WTS**

$$\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right)$$

Find a string $s$ such that

1. $|s| \geq p$
2. $s$ is in $L$
3. No matter how we cut $s$ into three (viable) pieces, some related string obtained by repeating the middle part falls out of $L$. 
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).

Consider the string

\[ s = 0^p1^p. \]

1. \(|s| \geq p \) ?
2. \( s \) is in \( L \) ?
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by repeating the middle part falls out of \( L \) ?
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof: Consider an arbitrary positive integer. WTS $p$ is not a pumping length for $L$. Consider the string $s = 0^p1^p$. Then, $s$ is in $L$ and $|s| = 2p \geq p$. Consider any division of $s$ into three parts $s = xyz$ with $|y|>0$, $|xy|\leq p$.

Since $|xy|\leq p$, $x = 0^k$, $y = 0^m$, $z = 0^r1^p$ with $k+m+r = p$, and since $|y| > 0$, $m>0$. Picking $i=0$: $xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p$, which is not in $L$ because $k+r < p$. Thus, no $p$ can be a pumping length for $L$ and $L$ is not regular.