Today's learning goals

- Explain what it means for a problem to be decidable.
- Justify the use of encoding.
- Give examples of decidable problems.
Recognition and enumeration  

1. Assume \( L \) is Turing-recognizable. WTS some enumerator enumerates it.

Let \( M \) be a TM that recognizes \( L \). We'll use \( M \) in a subroutine for high-level description of enumerator \( E \).

Idea: check each string in turn to see if it is in \( L = L(M) \).
Recognition and enumeration  

1. Assume L is Turing-recognizable. WTS some enumerator enumerates it.

Let M be a TM that recognizes L. We'll use M in a subroutine for high-level description of enumerator E.

Let \( s_1, s_2, \ldots \) be a list of all strings in \( \Sigma^* \) in standard string order

\( E = "\text{Ignore any input. Repeat the following for } i=1,2,3\ldots\"

1. Run M for (at most) \( i \) steps on each input \( s_1, \ldots, s_i \)
2. If any of the i computations of M accepts, print out the accepted string.

Correctness?
Example: union

Claim: The class of recognizable languages over fixed alphabet $\Sigma$ is closed under union.

Proof idea: Let $E_1$ and $E_2$ be enumerators. Construct the enumerator $E = "<ignore input>:

1. Run $E_1$ and $E_2$ in parallel
   i.e. for i=1, 2, 3...
   1a. Run $E_1$ for i steps.
   1b. Run $E_2$ for i steps.

   Whenever any string is printed by either enumerator, print it."
Suppose \( M \) is TM that recognizes \( L \)

<table>
<thead>
<tr>
<th>Suppose D is TM that decides ( L )</th>
<th>Suppose E is enumerator that enumerates ( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>If string ( w ) is in ( L ) then …</td>
<td>( M ) accepts ( w )</td>
</tr>
<tr>
<td></td>
<td>( D ) accepts ( w )</td>
</tr>
<tr>
<td></td>
<td>( E ) prints ( w ) (infinite time)</td>
</tr>
<tr>
<td>( \text{(WCM)} )</td>
<td>( M ) rejects ( w ) or ( M ) loops on ( w )</td>
</tr>
<tr>
<td></td>
<td>( D ) rejects ( w )</td>
</tr>
<tr>
<td></td>
<td>( E ) does not ever print ( w )</td>
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</table>
High-level description = Algorithm

Church-Turing thesis

Each algorithm can be implemented by some Turing machine.

"On input w
1. If |w| is even, accept.
2. Otherwise, form x= every second character in w."
Algorithms

So far: machines describing / recognizing sets.

What questions can we answer?
- Is a string a palindrome?
- Does a string have even length?

Answering these questions is the same as describing the set of strings for which the answer is yes.
Computational problems

A computational problem is **decidable** iff the language encoding the problem instances is **decidable**

- Does a specific DFA accept a given string? **encoded by** \{ representations of DFAs M and strings w such that w is in L(M) \}

- Is the language generated by a specific CFG empty? **encoded by** \{ representations of CFGs G such that L(G) = \(\emptyset\) \}

- Is a Turing machine a decider? **encoded by** \{ representations of Turing machines M such that M always halts \}
Representations for computational problems  

To decide these problems, we need to represent the objects of interest as **strings**

To define TM M:

"On input w ...

1. ...
2. ...
3. ...

For inputs that aren't strings, we have to **encode the object** (represent it as a string) first

**Notation:**

- \(<O>\) is the **string** that represents (encodes) the object \(O\)
- \(<O_1, ..., O_n>\) is the **single** string that represents the tuple of objects \(O_1, ..., O_n\)

Does a DFA accept \(\text{per} \cdot \text{string} \): input DFA + string \(<A, w>\)
To decide these problems, we need to represent the objects of interest as **strings**

To define TM M:

1. "On input w ..."
2. ...
3. ...

For inputs that aren’t strings, we have to **encode the object** (represent it as a string) first.

**Assumption:**
There are Turing machine subroutines that can decode the string representations of common objects so we can interact with them as intended.

E.g. from string representation of Turing machine, can decode \((Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})\)
Encoding inputs

**Payoff**: problems we care about can be reframed as languages of strings

e.g. "Recognize whether a string is a palindrome."
\[
\{ \text{w | w in } \{0,1\}^* \text{ and } w = w^R \}\]
e.g. "Check whether a string is accepted by a DFA."
\[
\{ \langle B, w \rangle | B \text{ is a DFA over } \Sigma, \ w \text{ in } \Sigma^*, \text{ and } w \text{ is in } L(B) \} \]
e.g. "Check whether the language of a PDA is infinite."
\[
\{ \langle A \rangle | A \text{ is a PDA and } L(A) \text{ is infinite} \} \]
Computational problems

Does a specific DFA accept a given string? encoded by
\{ representations of DFAs M and strings w such that w is in L(M) \}

Define using high-level description a Turing machine \( M_1 = \) "On input 
\langle B, w \rangle, where B is a DFA and w is a string:
1. Type-check encoding to check input is valid type.
2. Simulate B on input w (by keeping track of states in B, transition function of B, etc.)
3. If the simulations ends in an accept state of B, accept. If it ends in a non-accept state of B, reject. "

\( \varepsilon \)
Computational problems

• **Recall**: in high-level descriptions, can simulate (run) other Turing machines / algorithms as a subroutine of program being defined.

• **To prove decidable**: need to confirm that strings in the language are *accepted* and that strings not in the language are *rejected* (no looping allowed).
Define using high-level description a Turing machine $M_1 = \"On input $\langle B, w \rangle$, where $B$ is a DFA and $w$ is a string:\")

1. **Type check** encoding to check input is valid type.
2. Simulate $B$ on input $w$ (by keeping track of states in $B$, transition function of $B$, etc.)
3. If the simulations ends in an accept state of $B$, **accept**. If it ends in a non-accept state of $B$, **reject**.

**Why is $M_1$ a decider?**

ex. $M_1$ is answering $A \_{DFA}$
Computational problems

Vocabulary

\[ A_{xx} \] "Is a given string accepted by a given machine of type \( XX \)?"
\[ \{ <B,w> \mid B \text{ is a } XX \text{ over } \Sigma, w \in \Sigma^*, \text{ and } w \text{ is in } L(B) \} \]

\[ E_{xx} \] "Is the language of a machine of type \( XX \) is empty?"
\[ \{ <A> \mid A \text{ is a } XX \text{ over } \Sigma, L(A) \text{ is empty} \} \]

\[ EQ_{xx} \] "Are the languages of two given machines of type \( XX \) equal?"
\[ \{ <A, B> \mid A \text{ and } B \text{ are } XX \text{ over } \Sigma, L(A) = L(B) \} \]
For DFA

A. \( (A, 1) \) is in \( A_{DFA} \)
B. \( (A, 1) \) is in \( E_{DFA} \)
C. \( A \) is in \( E_{DFA} \)
D. \( (A, B) \) is in \( EQ_{DFA} \)
E. More than one of the above

- L(A) ≠ φ
- L(B) = ∅
Computational problems

For DFAs: which of the following computational problems are **decidable**?

A. $A_{\text{DFA}}$
B. $E_{\text{DFA}}$
C. $\text{EQ}_{\text{DFA}}$
D. All of the above
E. None of the above
Proving decidability

Claim: $E_{DFA}$ is decidable
Proof: WTS that $\{ <A> \mid A \text{ is a DFA over } \Sigma, L(A) \text{ is empty} \}$ is decidable.

Informally: what do you look for in the state diagram of a DFA to determine if it accepts *at least one* string?
Proving decidability

Claim: $E_{DFA}$ is decidable

Proof: WTS that \{ $<A>$ | A is a DFA over $\Sigma$, $L(A)$ is empty \} is decidable.

Test cases:

- e.g. $<$ is in $E_{DFA}$; $>$ is not in $E_{DFA}$

TM deciding $E_{DFA}$ should accept and should reject
Proving decidability

Claim: $E_{\text{DFA}}$ is decidable

Proof: WTS that $\{ <A> \mid A \text{ is a DFA over } \Sigma, L(A) \text{ is empty} \}$ is decidable.

Step 1: construction

*Idea: breadth-first search in state diagram to look for paths to F*
Proving decidability

Claim: $E_{\text{DFA}}$ is decidable

Proof: WTS that $\{ <A> | A \text{ is a DFA over } \Sigma, L(A) \text{ is empty} \}$ is decidable.

Step 1: construction

Idea: breadth-first search in state diagram to look for paths to $F$

Define TM $M_2$ by: $M_2 = "\text{On input } <A>:"

1. Check whether input is a valid encoding of a DFA; if not, reject.
2. Mark the start state of $A$.
3. Repeat until no new states get marked:
   i. Loop over states of $A$ and mark any unmarked state that has an incoming edge from a marked state.
4. If no final state of $A$ is marked, accept; otherwise, reject."
Proving decidability

Step 2: correctness proof

WTS (1) $L(M_2) = E_{DFA}$ and (2) $M_2$ is a decider.
For next time

GroupHW5 due Saturday, May 19

For Friday, pre-class reading: Section 4.1, Theorem 4.5 (page 197) and Theorem 4.8 (page 199)