Today's learning goals

- Design TMs using different levels of descriptions.
- Determine whether a Turing machine is a decider.
- Prove properties of the classes of recognizable and decidable sets.
Describing TMs

- **Formal definition**: set of states, input alphabet, tape alphabet, transition function, state state, accept state, reject state.

- **Implementation-level definition**: English prose to describe Turing machine head movements relative to contents of tape.

- **High-level description**: Description of algorithm, without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.
Language of a TM

$L(M) = \{ w \mid M \text{ accepts } w \}$

If $w$ is in $L(M)$ then the computation of $M$ on $w$ halts and accepts.

If the computation of $M$ on $w$ halts and rejects, then $w$ is not in $L(M)$.

If the computation of $M$ on $w$ doesn't halt, then $w$ is not in $L(M)$
Deciders and recognizers \textit{Sipser} p. 170 Defs 3.5 and 3.6

- L is \textbf{recognized} by Turing machine M if \( L(M) = M \).

- M is a \textbf{decider} if it is a Turing machine and halts on all inputs.

- L is \textbf{decided} by Turing machine M if M is a decider and \( L(M) = L \).
An example

Which of the following is an implementation-level description of a TM which decides the empty set?

M = "On input w:

A. reject."

B. sweep right across the tape until find a non-blank symbol. Then, reject."

C. If the first tape symbol is blank, accept. Otherwise, reject."

D. More than one of the above.

E. I don't know."
Extension

- Give an implementation-level description of a Turing machine which recognizes (but does not decide) the empty set.

- Give a high-level description of this Turing machine.
Another example

Suppose $M_1$ and $M_2$ are Turing machines. Consider the new TM $M =$ "On input $w$,
1. Run $M_1$ on $w$. If $M_1$ rejects, rejects. If $M_1$ accepts, go to 2.
2. Run $M_2$ on $w$. If $M_2$ accepts, accept. If $M_2$ rejects, reject."

What kind of construction is this?
A. Formal definition of TM
B. Implementation-level description of TM
C. High-level description of TM
D. I don't know.

What's $L(M)$?
Is $M$ a decider?
Classifying languages

A language $L$ is

**Turing-recognizable** if there is a TM $M$ such that $L(M) = L$ in other words, if there is some TM that recognizes it.

**Turing-decidable** if there is a TM $M$ such that $M$ is a decider and $L(M) = L$ in other words, if there is some TM that decides it.
Turing recognizable languages

Turing decidable languages

Context-free languages

Regular languages
Closure

**Theorem:** The class of decidable languages over fixed alphabet $\Sigma$ is closed under union.

Proof: Let …

WTS …
Closure

**Theorem:** The class of decidable languages over fixed alphabet $\Sigma$ is closed under union.

Proof: Let $L_1$ and $L_2$ be languages over $\Sigma$ and suppose $M_1$ and $M_2$ are TMs deciding these languages. We will define a new TM, $M$, via a high-level description. We will then show that $L(M) = L_1 \cup L_2$ and that $M$ always halts.
Closure

**Theorem:** The class of decidable languages over fixed alphabet $\Sigma$ is closed under union.

Proof: Let $L_1$ and $L_2$ be languages and suppose $M_1$ and $M_2$ are TMs deciding these languages. Construct the TM $M$ as "On input $w$,

1. Run $M_1$ on input $w$. If $M_1$ accepts $w$, accept. Otherwise, go to 2.
2. Run $M_2$ on input $w$. If $M_2$ accepts $w$, accept. Otherwise, reject."

Correctness of construction:

WTS $L(M) = L_1 \cup L_2$ and $M$ is a decider.
**Closure**

Good exercises — can’t use without proof! (Sipser 3.15, 3.16)

<table>
<thead>
<tr>
<th>The class of decidable languages is closed under</th>
<th>The class of recognizable languages is closed under</th>
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<tbody>
<tr>
<td>- Union ✓</td>
<td>- Union</td>
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<tr>
<td>- Concatenation</td>
<td>- Concatenation</td>
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<td>- Intersection</td>
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<td>- Complementation</td>
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For next time

**Group HW4 due** Saturday, May 12

For Monday, pre-class reading: Section 3.2, pp. 181