1. (3 points) Prove or give a counterexample ($\phi$ and $\psi$ are propositional logic formulas):

   1. If $(\phi \rightarrow \psi)$ is valid and $\phi$ is valid, then $\psi$ is valid.
   
   2. If $(\phi \rightarrow \psi)$ is satisfiable and $\phi$ is satisfiable, then $\psi$ is satisfiable.
   
   3. If $(\phi \rightarrow \psi)$ is valid and $\phi$ is satisfiable, then $\psi$ is satisfiable.

2. (i) (2 points) Let the binary connective $\downarrow$ mean “nor” (i.e., $p \downarrow q$ is true iff neither $p$ nor $q$ are true). Show that $\downarrow$ is functionally complete, i.e. for every truth table $T$ there is a formula built from the propositional variables of $T$ and the connective $\downarrow$, whose truth table is $T$.

   (ii) (4 points) Show that the set of connectives $\{\leftrightarrow, \neg\}$ is not functionally complete.

3. (4 points) For a propositional formula $\varphi$, let $P_\varphi$ denote the set of propositional symbols used in $\varphi$. Suppose $\varphi$ and $\psi$ are formulas, not necessarily over the same set of propositional symbols. We say that $\varphi \models \psi$ iff for every truth assignment $\theta$ for $P_\varphi \cup P_\psi$, if $\theta \models \varphi$ then $\theta \models \psi$. Prove that, if $\varphi \models \psi$, then there exists a formula $\xi$ such that $P_\xi \subseteq P_\varphi \cap P_\psi$, $\varphi \models \xi$, and $\xi \models \psi$.

4. (5 points) König’s Lemma says that any infinite, finitely branching tree has an infinite path. Use König’s Lemma to provide an alternative (simpler!) proof of the Compactness theorem for propositional logic.

5. (4 points) Prove the following version of De Morgan’s Laws using natural deduction:

   $$\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$$

6. (3 points) Use resolution to show that the formula

   $$((r \rightarrow (p \land q)) \rightarrow ((r \rightarrow p) \land (r \rightarrow q)))$$

   is valid.