Lecture Unit 8: Elementary Set Theory
**Set operations: Basic definitions**

**Definition: Subset and proper subset**

\[
A \subseteq B \iff \forall x, \text{ if } x \in A \text{ then } x \in B
\]

\[
A \nsubseteq B \iff \exists x, \text{ such that } x \in A \text{ and } x \notin B
\]

\[
A \subset B \iff (A \subseteq B) \land (\exists x \in B \text{ such that } x \notin A)
\]

**Definition: Set equality**

\[
A = B \iff (A \subseteq B) \land (B \subseteq A)
\]

**Definition: Set intersection and union**

\[
A \cap B = \{x : x \in A \text{ and } x \in B\}
\]

\[
A \cup B = \{x : x \in A \text{ or } x \in B\}
\]
Definition: *Set difference and symmetric difference*

\[ A - B = A \setminus B = \{ x : x \in A \text{ and } x \notin B \} \]

\[ A \oplus B = \{ x : (x \in A \land x \notin B) \text{ or } (x \in B \land x \notin A) \} \]

Note:

\[ A \oplus B = (A - B) \cup (B - A) \]

Definition: *Cartesian product of sets*

\[ A \times B = \{ (x,y) : x \in A \text{ and } y \in B \} \]

More generally:

\[ \prod_{i=1}^{k} A_i = \{ (x_1, x_2, \ldots, x_k) : x_1 \in A_1, x_2 \in A_2, \ldots, x_k \in A_k \} \]

\[ A^k = \{ (x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in A \} \]
Algebraic rules for sets

In the table below $P, Q,$ and $R$ are arbitrary subsets of the same universal set $U$; all the complements are with respect to $U$.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Equation 1</th>
<th>Equation 2</th>
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</thead>
<tbody>
<tr>
<td>Commutative Rule</td>
<td>$P \cap Q = Q \cap P$</td>
<td>$P \cup Q = Q \cup P$</td>
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<tr>
<td>Associative Rule</td>
<td>$(P \cap Q) \cap R = P \cap (Q \cap R)$</td>
<td>$(P \cup Q) \cup R = P \cup (Q \cup R)$</td>
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<tr>
<td>Distributive Rule</td>
<td>$(P \cap Q) \cup (P \cap R) = P \cap (Q \cup R)$</td>
<td>$(P \cup Q) \cap (P \cap R) = P \cup (Q \cap R)$</td>
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<tr>
<td>Idempotent Rule</td>
<td>$P \cap P = P$</td>
<td>$P \cup P = P$</td>
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<td>Absorption Rule</td>
<td>$P \cup (P \cap Q) = P$</td>
<td>$P \cap (P \cup Q) = P$</td>
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<tr>
<td>DeMorgan Rule</td>
<td>$(P \cap Q)^c = P^c \cup Q^c$</td>
<td>$(P \cup Q)^c = P^c \cap Q^c$</td>
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</tbody>
</table>
Venn diagrams for sets are similar to truth tables in propositional logic. They describe all possible intersections between several sets.

In general, a Venn diagram for \( n \) sets will have \( 2^n \) distinct regions, similar to minterms in logic. For example:

\[
P Q^c R^c = \{ x \in U : (x \in P) \wedge (x \in Q^c) \wedge (x \in R^c) \}\]
Venn diagrams for 4 and 5 sets

Venn diagrams for sets are similar to truth tables in propositional logic. They describe all possible intersections between several sets.

Beyond five sets, Venn diagrams become rather unwieldy.
Example: Prove that \( Q \cap (P \cap R)^c = Q \cap (P \cap Q \cap R)^c \)

\[
\begin{align*}
P \cap Q \cap R &= \{8\} \\
(P \cap Q \cap R)^c &= \{1, 2, 3, 4, 5, 6, 7\} \\
Q &= \{2, 5, 6, 8\}
\end{align*}
\]

Therefore:
\[
Q \cap (P \cap Q \cap R)^c = \{2, 5, 6\}
\]

Now:
\[
\begin{align*}
P \cap R &= \{7, 8\} \\
(P \cap R)^c &= \{1, 2, 3, 4, 5, 6\} \\
Q &= \{2, 5, 6, 8\}
\end{align*}
\]

Therefore:
\[
Q \cap (P \cap R)^c = \{2, 5, 6\} = Q \cap (P \cap Q \cap R)^c
\]
Pascal Triangle

**Pascal triangle** is a table and method to systematically compute the binomial coefficients \(^n\choose k\) for all \(n\) and \(k\).

**Initialization:**

\[
\begin{align*}
\binom{0}{0} &= \binom{n}{0} = \binom{n}{n} = 1
\end{align*}
\]

**Recursion:**

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

This triangle was described by **Blaise Pascal** in his *Traité du Triangle Arithmétique* (1653), though it was known to ancient Hindus, Greeks, and Chinese many centuries earlier (2-nd century BC).
Set partitions

Let \( A \) be an arbitrary set. A **set partition** of \( A \) is a set \( \Omega \subset P(A) \) with the following properties:

1. For all \( X \in \Omega \), we have \( X \neq \emptyset \)
2. For all \( X, Y \in \Omega \), either \( X = Y \) or \( X \cap Y = \emptyset \)
3. The union of all \( X \in \Omega \) is \( A \) itself: \( A = \bigcup_{X \in \Omega} X \)

The elements of a partition \( \Omega \subset P(A) \) are called **blocks**. Note that blocks are themselves sets; they are subsets of the original set \( A \).

**Example:**

\[ A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \]

\[ \Omega = \{\{1\}, \{2\}, \{3, 5\}, \{4, 7\}, \{6, 8, 10\}, \{9, 11, 12\}\} \]
Counting partitions: Bell numbers

The total number of different partitions of a set of size \( n \) is known as the **Bell number** \( B(n) \), defined as follows:

\[
B(n) = S(n, 1) + S(n, 2) + \cdots + S(n, n-1) + S(n, n)
\]

The Bell numbers \( B(n) \) and the Stirling numbers \( S(n, k) \) can be computed recursively, in a manner similar to the Pascal triangle.

### Initialization:

\[
S(n, 1) = S(n, n) = 1
\]

### Recursion:

\[
S(n, k) = S(n-1, k-1) + kS(n-1, k)
\]

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<th>5</th>
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