Lecture Unit 3: Predicate Logic (with examples from number theory)
Let $D = \{1,2,3,4\}$ and let $P(x)$ be the predicate “$x^2 > x$”. Then:

\[ \forall x \in D, P(x) \] means “For all $x$ in the set $\{1,2,3,4\}$, we have $x^2 > x$”

Let $D = \mathbb{R}$, the set of real numbers, and let $P(x)$ be “$x^2 > 1$”. Then:

\[ \exists x \in D, P(x) \] is True

\[ \forall x \in D, P(x) \] is False

\[ \exists! x \in D, P(x) \] is False

Let $D = \text{the set of students in CSE20}$, let $P(x)$ be “$x$ studies hard”, and let $Q(x)$ be “$x$ gets an A”. Then:

\[ \forall x \in D, P(x) \rightarrow Q(x) \] means:

“All students in CSE20 who study hard will get an A”
Here are some useful sets of numbers that will be often used as domains throughout this course:

<table>
<thead>
<tr>
<th>Set</th>
<th>Symbol</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real numbers</td>
<td>$\mathbb{R}$</td>
<td>-1729, 0, $\frac{3}{4}$, 1, $e$, $\sqrt[3]{31}$, $\pi$, 196883</td>
</tr>
<tr>
<td>Rational numbers</td>
<td>$\mathbb{Q}$</td>
<td>ratios of integers: -1, 42, $\frac{3}{4}$, $\frac{5}{17}$</td>
</tr>
<tr>
<td>Irrational numbers</td>
<td>$\mathbb{R}\setminus\mathbb{Q}$</td>
<td>$\sqrt{2}$, $\pi$, $\sqrt[3]{7}$, $\log_2{3}$, $(1 + \sqrt{5})/2$</td>
</tr>
<tr>
<td>Integers</td>
<td>$\mathbb{Z}$</td>
<td>${\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots }$</td>
</tr>
<tr>
<td>Natural numbers</td>
<td>$\mathbb{N}$</td>
<td>${0, 1, 2, 3, 4, 5, 6, 7, \ldots }$</td>
</tr>
<tr>
<td>Positive integers</td>
<td>$\mathbb{Z}^+ = \mathbb{N}^+$</td>
<td>${1, 2, 3, 4, 5, 6, 7, 8, \ldots }$</td>
</tr>
<tr>
<td>Primes</td>
<td>$\mathbb{P}$</td>
<td>${2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots }$</td>
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</tbody>
</table>
Let $P(x)$ be the predicate “$x > 2$” and $Q(x)$ be the predicate “$x < x^2$”, both defined on the domain $D = \mathbb{R}$. Express in predicate logic the following statement:

“There is a real number greater than 2 that is strictly smaller than its square”

A: $\exists x \in D, \ P(x) \rightarrow Q(x)$

B: $\exists x \in D, \ P(x) \lor Q(x)$

C: $\exists x \in D, \ Q(x) \rightarrow P(x)$

D: $\exists x \in D, \ P(x) \land Q(x)$

E: None of the above
Let $P(x)$ be the predicate “$x > 2$” and $Q(x)$ be the predicate “$x < x^2$”, both defined on the domain $D = \mathbb{R}$. Express in predicate logic the following statement:

“Every real number greater than 2 is strictly smaller than its square”

A: $\forall x \in D, P(x) \rightarrow Q(x)$ ✓

B: $\forall x \in D, P(x) \lor Q(x)$ ✗

C: $\forall x \in D, Q(x) \rightarrow P(x)$ ✗

D: $\forall x \in D, P(x) \land Q(x)$ ✗

E: None of the above
Let $P(x)$ and $Q(x)$ be predicates defined on some common domain $D$. What is the **negation** of the statement:

$$\forall x \in D, P(x) \rightarrow Q(x)$$

- **A:** $\forall x \in D, P(x) \rightarrow \sim Q(x)$
- **B:** $\exists x \in D, P(x) \land \sim Q(x)$
- **C:** $\exists x \in D, P(x) \rightarrow \sim Q(x)$
- **D:** $\forall x \in D, P(x) \land \sim Q(x)$
- **E:** None of the above
Example: Consider the following implication statement:

“If a natural number \( n \) is prime, then \( n \) is odd or \( n = 2 \)”

Write down the contrapositive, the inverse, and the converse of this implication statement. Which are true and which are false?

Solution: Define predicate \( P(n) \) as “\( n \) is prime” and the predicate \( Q(n) = Q_1(n) \lor Q_2(n) \), where \( Q_1(n) \) is “\( n \) is odd” and \( Q_2(n) \) is “\( n = 2 \)”. Then:

\[
\text{Implication: } \forall x \in \mathbb{N}, P(n) \rightarrow Q(n)
\]

Contrapositive: \( \forall x \in \mathbb{N}, \neg Q(n) \rightarrow \neg P(n) \)

If a natural number \( n \) is even and \( n \neq 2 \), then \( n \) is not prime.

Inverse: \( \forall x \in \mathbb{N}, \neg P(n) \rightarrow \neg Q(n) \)

If a natural number \( n \) is not prime, then \( n \) is even and \( n \neq 2 \).

Converse: \( \forall x \in \mathbb{N}, Q(n) \rightarrow P(n) \)

If a natural number \( n \) is odd or \( n = 2 \), then \( n \) is prime.