1. (a) First note that all the strings in the set $A \times B \times A$ are of the same length, which greatly simplifies the lexicographic order. Now, here is how to find the successor of $((1, 0), c, (1, 1))$. Since $(1, 1)$ is the last element in $A$, we need to increase $c$. But since $c$ is the last element in $B$, we have to increase $(1, 0)$ to $(1, 1)$ instead. The first string that starts with $(1, 1)$ is $((1, 1), a, (0, 0))$, so the correct answer is (2).

(b) In each case, the domain and range of $f$ are specified: the domain is $A = 5 = \{1, 2, 3, 4, 5\}$ and the range is $B = 4 = \{1, 2, 3, 4\}$, except in part (b) where the range is $B = 5$.

   a. The function is not completely specified, since we do not know which values in $B$ it takes. The function is not an injection since its coimage is not the trivial partition of $A$. It is also not a surjection, since it takes on only two out of the four different values in $B$.

   b. Once again, the function is not completely specified, since we do not know which value to assign to each element of $A$. However, we know that $f$ is an injection since its coimage is the trivial partition of $A$. We also know that it is a surjection since $|\text{Coimage}(f)| = |\text{Image}(f)| = |B|$. It follows that $f$ is a bijection. In fact, since $A = B$, this function is a permutation on $A$.

   c. The situation here is the same as in part (a), except that we do know the value of $f(x)$ for each $x \in A$. Specifically, $f(x) = 2$ if $x \in \{1, 3, 5\}$ and $f(x) = 4$ if $x \in \{2, 4\}$. Since $\{1, 3, 5\} \cup \{2, 4\} = A$, this is all we need. As in part (a), this function is neither an injection nor a surjection.

   d. Since $|\text{Image}(f)| = |B|$, this function is a surjection. Since $|A| > |B|$, it cannot be an injection. Clearly, it is not completely specified.

   (c) First, let us determine whether the statement is true. To this end, we proceed as follows:

   $$f^{-1}(C \cap D^c) = \{x \in X : f(x) \in C \cap D^c\}$$
   $$= \{x \in X : f(x) \in C\} \cap \{x \in X : f(x) \in D^c\}$$
   $$= \{x \in X : f(x) \in C\} \cap \left(\{x \in X : f(x) \in D\}\right)^c$$
   $$= f^{-1}(C) \cap (f^{-1}(D))^c$$

   So the statement is true. Now observe that for any two sets $A$ and $B$, it is true that $A \cap B^c = A - B$. Hence $C \cap D^c = C - D$ and $f^{-1}(C) \cap (f^{-1}(D))^c = f^{-1}(C) - f^{-1}(D)$. It follows that the correct answer is (0).

2. (a) The statement is false. For example, take $A = \{1\}$ and $B = C = \{2\}$. Then $A \nsubseteq B$ and $B \cap C = \{2\} \neq \emptyset$. Yet $A \cap C = \emptyset$.

   (b) The statement is true, and here is a proof. Using the associativity of the intersection and the idempotent rule $B^c \cap B^c = B^c$, we have:

   $$(C - B) \cap (A - B) = (C \cap B^c) \cap (A \cap B^c) = A \cap C \cap B^c$$

   If $(A \cap C) \subseteq B$, then no element of $A \cap C$ belongs to $B^c$, and hence $A \cap C \cap B^c = \emptyset$.

(c) Here $A - B \cup (A \cap B) = A$, which can be seen as follows:

   $$(A - B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B) = A \cap (B^c \cup B) = A \cap U = A$$

   where we have used the distributive law, along with the fact that $B^c \cup B$ is the universal set $U$ for any set $B$, and $A \cap U = A$ for any set $A$. 
(d) Once again, we have \((A \setminus (A \cap B)) \cup (A \cap B) = A\), which can be seen as follows. First, observe that:

\[
A \setminus (A \cap B) = A \cap (A \setminus B) = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = A \cap B^c
\]

Observe that \(A \cap B^c = A - B\). Hence \((A \setminus (A \cap B)) \cup (A \cap B) = (A - B) \cup (A \cap B)\), which is precisely the set expression we have simplified in part (c).

3. The first 64 rows of the Pascal triangle, reduced modulo 2, are:

In case you’re interested in fractals, the first 64 rows of the Pascal triangle, reduced modulo 2, are a pretty good approximation of the Sierpinski triangle. This fractal was originally thought up by Waclaw Sierpinski in the late 1930s. It predated the Mandelbrot set, usually considered the “first” fractal. Computing the Pascal triangle modulo 2, is an efficient way to generate the Sierpinski triangle. In fact, here is the simple loop (in C language) used to generate the output on this page.
/* Allocate the Pascal/Sierpinski triangle */

max_n = 64;
Pascal = Allocate_Bit_Matrix(max_n,max_n);

/* Compute the Pascal/Sierpinski triangle */

Pascal[0][0] = 1;
for (n = 1; n < max_n; n++)
{
    Pascal[0][n] = 1;
    for (k = 1; k < n; k++)
        Pascal[k][n] = (Pascal[k][n-1] + Pascal[k-1][n-1]) % 2;
    Pascal[n][n] = 1;
}

Note that if you try to work with actual binomial coefficients \( \binom{n}{k} \) and reduce modulo 2 only in the end, you are likely to encounter overflow (and other) problems in software since \( \binom{n}{k} \) is quite large when \( n = 63 \). One of the goals in this problem was to have you realize how to compute what you need efficiently using the principles of modular arithmetic.

4. Starting with the general recurrence \( S(n,k) = S(n-1,k-1) + kS(n-1,k) \), we obtain, for the special case \( k = 2 \), the following:

\[
S(n,2) = S(n-1,1) + 2S(n-1,2) = 2S(n-1,2) + 1
\]

Let \( a_n = S(n,2) \), and note that \( a_2 = S(2,2) = 1 \). Then, using the recurrence \( a_n = 2a_{n-1} + 1 \), we can readily compute:

\[
a_3 = 3, \quad a_4 = 7, \quad a_5 = 15, \quad a_6 = 31, \quad a_7 = 63, \quad a_8 = 127, \quad \cdots
\]

which leads to the conjecture that \( a_n = S(n,2) = 2^{n-1} - 1 \) for all \( n \geq 2 \). This conjecture can be easily proved by induction (verify this!). Alternatively, we can count \( S(n,2) \) directly, as follows. We need to partition the \( n \) elements in a set \( X \) into exactly two subsets, say \( A \) and \( B \). For each element \( x \in X \), we have 2 choices: either put it in \( A \) or put it in \( B \). This gives altogether \( 2^n \) choices, but also includes the cases \( A = \emptyset, B = X \) and \( A = X, B = \emptyset \), which are not valid partitions. Thus the number of valid choices is \( 2^n - 2 \). However, observe that in this counting argument, each partition is counted twice: once as \( \{A, B\} \) and once as \( \{B, A\} \). Thus the total number of partitions is \( S(n,2) = (2^n - 2)/2 = 2^{n-1} - 1 \).

5. A relation \( Q \) from \( A \) to \( B \) is functional if and only if \( \forall a \in A, \exists! b \in B, \text{ such that } (a, b) \in Q \). The relation \( R \) is clearly not functional since for \( 4 \in A \), there is no \( b \in B \) with \( (4, b) \in R \) (this relation also fails the uniqueness test, why?). The relation \( S \) is also not functional since \( (5, 5) \in S \) and \( (5, 7) \in S \), which fails the uniqueness test. This relation becomes functional if we remove either 5 or 7 from \( B \). Finally, it is easy to see that \( T \) is not functional as well. It would become functional if we replace the pair \( (6, 5) \) with \( (5, 5) \), for example.
6. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$, let $g: \mathcal{Y} \rightarrow \mathcal{Z}$, and let $h = g \circ f$ be the composition function from $\mathcal{X}$ to $\mathcal{Z}$.

(a) If $h$ is an injection, then $f$ must also be an injection. We will prove the contrapositive: if $f$ is not an injection, then neither is $h$. Indeed, suppose that $f(x_1) = f(x_2) = y$ for some distinct $x_1, x_2 \in \mathcal{X}$ (so that $f$ is not an injection). Then $h(x_1) = g(f(x_1)) = g(y)$ and $h(x_2) = g(f(x_2)) = g(y)$. Thus $h(x_1) = h(x_2)$ for $x_1 \neq x_2$; so $h$ is not an injection.

If $h$ is an injection, then the function $g$ need not be an injection. For example, take the sets $\mathcal{X} = \{1, 2\}$, $\mathcal{Y} = \{2, 4, 8\}$, and $\mathcal{Z} = \{1, 2\}$. Let $f(x) = 2^x$ for all $x \in \mathcal{X}$, and let $g(y) = y \mod 3$ for all $y \in \mathcal{Y}$. Then $h$ is an injection, but $g$ is not, since $g(2) = g(8)$.

(b) If $h$ is a surjection, then $g$ must also be a surjection. Note that given $z \in \mathcal{Z}$, we can always find $y \in \mathcal{Y}$, such that $g(y) = z$. Indeed, because $h$ is a surjection, there exists $x \in \mathcal{X}$, such that $h(x) = g(f(x)) = z$. Since $f(x) \in \mathcal{Y}$, we can always take $y = f(x)$.

If $h$ is a surjection, then the function $f$ need not be a surjection. As a counter-example, consider the example from part (a). In this example, $h$ is a surjection, but $f$ is not.

(c) If $h$ is a bijection, then we know from parts (a) and (b) that $f$ is an injection and $g$ is a surjection. However, $f$ need not be a surjection and $g$ need not be an injection. For instance, in the example of part (a), it is easy to see that $h$ is actually a bijection. However, we have already seen that $f$ is not a surjection and $g$ is not an injection in this example. Thus neither $f$ nor $g$ need to be bijections.