1. (a) Which of the following statements is true:
   
   (0) A number \( k \) divides the sum of three consecutive integers \( n, n + 1, \) and \( n + 2 \) if and only if it divides the middle integer \( n + 1 \).
   
   (1) An integer \( n \) is divisible by 6 if and only if it is divisible by 3.
   
   (2) For all integers \( a, b, \) and \( c, a \mid bc \) if and only if \( a \mid b \) and \( a \mid c \).
   
   (3) For all integers \( a, b, \) and \( c, a \mid (b + c) \) if and only if \( a \mid b \) and \( a \mid c \).
   
   (4) If \( r \) and \( s \) are integers, then \( r \mid s \) if and only if \( r^2 \mid s^2 \).

   (b) Prove that for all odd integers \( n \), we have \( n^4 = 1 \pmod{16} \).

   (c) To one percent accuracy, the number of integers \( n \) in the list \( 0^4, 1^4, 2^4, \ldots, 1000^4 \) such that \( n \mod 16 = 1 \) is:
   
   (0) 20 percent
   
   (1) 50 percent
   
   (2) 30 percent
   
   (3) 35 percent
   
   (4) 25 percent

   (d) Prove directly from the definition of the least common multiple (LCM) that \( a \mid b \) if and only if \( \text{lcm}(a, b) = b \).

In parts (b), (d), you need to prove the given statements. In parts (a), (e), you must not only indicate the correct answer, but also explain in detail how you arrived at this result.

2. It is known that an integer \( a \) divides the sum and the difference of two integers \( n \) and \( m \), namely \( a \mid (n + m) \) and \( a \mid (n - m) \). Does it follow that \( a \) divides \( n \), if it is also known that:

   (a) \( a \) is even?
   
   (b) \( a \) is odd?

3. Prove that \( \sqrt{2} + \sqrt{6} \) is irrational. (Note that, in general, the sum of two irrational numbers could be rational.)

4. This problems concerns modular arithmetic, in particular the integers modulo 3.

   (a) Prove that for all integers \( n \), we have either \( n^2 = 0 \pmod{3} \) or \( n^2 = 1 \pmod{3} \).
   
   (b) Let \( m, n \in \mathbb{Z} \). Prove that if 3 divides \( m^2 + n^2 \), then 3 divides \( m \) and 3 divides \( n \).

5. (a) Using the Euclidean algorithm, compute the greatest common divisor of 3510 and 672.

   (b) Backtrace the computation in the Euclidean algorithm to find a pair of integers \( a \) and \( b \) such that \( \gcd(3510, 672) = 3510a + 672b \).
6. You are given a fair balance, consisting of a pair of scales. Each time the balance is used, either the right scale will tip (if the weight on the right is strictly greater than the weight on the left), or the left scale will tip, or the two scales will be in balance (if the two weights are exactly equal). You are also given an infinite supply of two types of weights: small weights, each weighing \( m \) ounces, and large weights, each weighing \( n \) ounces.

Finally, you also have a golden crown that weighs exactly \( x \) ounces, where \( x \) is a positive integer. Your goal is to determine the weight of the crown.

(a) Suppose that \( m = 100 \) and \( n = 441 \). Can you determine the weight of the crown? If yes, explain why. If not, explain why not.

(b) Now suppose that \( m = 6 \) and \( n = 9 \). Can you determine the weight of the crown? If yes, explain why. If not, explain why not.

(c) Extra credit: (5 points). Can the crown’s weight be determined if \( m = 6 \) and \( n = 88 \)?