INSTRUCTIONS

The exam consists of six problems worth a total of 130 points. Of these, 30 points in Problem 6 are extra-credit points. That is, a score of only 100 points is needed to receive the full 25% credit given to the midterm exam. Scores in excess of 100 points will compensate for the final exam and/or homework scores.

Write your answers in the spaces provided. Show all your work, except in the true/false problem. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress.

Good luck!
Problem 1 (12 points)

In the following statements, \( \mathbb{P} \) is the set of primes, \( \mathbb{N} \) is the set of natural numbers, and \( \mathbb{Z} \) is the set of integers. Which of these statements are true and which are false?

True  False

☐  ☐ The statement form \( pq \lor \neg p \lor \neg q \) is a tautology.

☐  ☐ The statement forms \( \sim (p \lor (q \land r)) \) and \( \neg p \lor \neg q \lor \neg r \) are logically equivalent.

☐  ☐ \( \forall n \in \mathbb{Z}, \text{ if } 3n + 2 \text{ is even then } n + 5 \text{ is odd.} \)

☐  ☐ \( \exists m \in \mathbb{N}, \forall n \in \mathbb{Z}, \text{ if } n \geq m \text{ then } (n^2 - n + 17) \in \mathbb{P}. \)

☐  ☐ The last decimal digit of the number \( 42^7 \) is 7.

☐  ☐ The number 10110100101100\(_2\), expressed in binary, is divisible by 4.

No justification is required. Grading: for each of the six statements above, 2 points if the correct box is the only one checked, 1 point if no box is checked, and 0 points in all other cases.
Problem 2 (18 points)

a. The numbers $x = 210_3$ and $y = 112_3$ are expressed in the ternary (base-3) number system. Compute their sum $a = x + y$ and product $b = xy$ using ternary arithmetic. Show all stages of your ternary computation and express its results in the ternary number system.

\[
\begin{align*}
a &= 3 \\
b &= 3
\end{align*}
\]
b. In this problem, \( n \) is an arbitrary integer with \( n \geq 2 \). What is \( (n - 1)^2(n + 1) \) modulo \( n \)?

\[
(n - 1)^2(n + 1) = \mod n
\]

c. Let \( m = 2^{11} \cdot 5 \cdot 7^2 \cdot 11 \) and \( n = 3^7 \cdot 5^3 \cdot 7^2 \). Let \( d \) be the largest integer such that \( d \mid m \) and \( d \mid n \). What is the prime factorization of \( d \)?

\[
d =
\]
Problem 3 (30 points)

This problem deals with three Boolean variables $p$, $q$, and $r$, defined as follows:

$p$ takes the value 1 if there is a power outage, and takes the value 0 otherwise;
$q$ takes the value 1 if there is a quake, and takes the value 0 otherwise;
$r$ takes the value 1 if there is a riot, and takes the value 0 otherwise.

The goal of this problem is to design and implement the Boolean function $f(p, q, r)$ that evaluates to 1 (and then wakes-up Governor Brown!) if and only if at least two of the three calamities (power outage, quake, riot) occur.

a. Write down the truth table for the Boolean function $f(p, q, r)$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$f(p, q, r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

b. Express the function $f(p, q, r)$ in disjunctive normal form (DNF).

$$f(p, q, r) =$$
c. Express the function \( f(p, q, r) \) in conjunctive normal form (CNF).

\[
\begin{aligned}
f(p, q, r) &= \\
\end{aligned}
\]

d. Design a logic circuit that implements this function. You can use any number of NOT gates, and any number of two-input AND, OR, XOR gates to implement the function. However, your score will decrease by 2 points for each gate in your design beyond the minimum number of gates required to implement this function.

*Hint:* Recall the carry in a full-adder.
Problem 4 (20 points)

Let \( \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots\} \) be an infinite sequence of real numbers. According to the usual mathematical definition of convergence, we say that the sequence \( \{a_n\}_{n=1}^{\infty} \) converges to zero if:

For any positive real number \( \varepsilon \), there exists a positive integer \( m \), such that for all integers \( n \geq m \), it is true that \( |a_n| < \varepsilon \).

For the purposes of this problem, define the following domains and predicates:

\[
\begin{align*}
\mathbb{R}^+ & \triangleq \text{the set of positive real numbers} \\
\mathbb{Z}^+ & \triangleq \text{the set of positive integers} \\
P(n, m) & \triangleq \text{"} n \geq m \text{"} \\
Q(a, \varepsilon) & \triangleq \text{"} |a| < \varepsilon \text{"}
\end{align*}
\]

a. Consider the infinite sequence \( \{a_n\}_{n=1}^{\infty} \) defined by \( a_n = 1/n \). Using the domains \( \mathbb{R}^+ \) and \( \mathbb{Z}^+ \) along with the predicates \( P(n, m) \) and \( Q(a, \varepsilon) \), express the fact that this sequence converges to zero in predicate logic. You can use the implication \( \rightarrow \) connective; however, you should not use any other domains or predicates.

\( S: \)
b. Let $S$ be the statement from part (a). What is the contrapositive, converse, inverse, and negation of this statement? Express your answers in predicate logic — do not write them out in words. However, in this part, you can no longer use the implication $\rightarrow$ connective.

Contrapositive:

Converse:

Inverse:

Negation:
Problem 5 (20 points)

a. Let \( n \) be a positive even integer. What is the remainder when \( n^2 \) is divided by 4?

\[ n^2 = \mod 4 \]

b. Let \( n \) be a positive odd integer. What is the remainder when \( n^2 \) is divided by 4?

\[ n^2 = \mod 4 \]
c. Let $m = 4k + 3$, where $k$ is a positive integer. Show that $m$ cannot be expressed as the sum of squares of two integers. That is, there do not exist $a, b \in \mathbb{Z}$ such that $m = a^2 + b^2$.

*Hint:* Use what you found in parts (a) and (b) of this problem.
Extra-credit Problem 6 (30 points)

Consider the following statement: There exists a unique prime $p$, such that $p + 2$ and $p + 4$ are also primes. Prove this statement, or, if you think that it is false, prove that it is false.