Randomized Algorithms, Hash Functions

Russell Impagliazzo and Miles Jones
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http://cseweb.ucsd.edu/classes/sp16/cse21-bd/
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Selection Problem: WHAT

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

find the $i^{th}$ smallest element in the array.

effectively find the entry $x$ such that

$i-1$ elements are smaller than $x$
Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

find the $i^{th}$ smallest element in the array.

What algorithm would you choose if $i=1$?
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

find the $i^{th}$ smallest element in the array.

What algorithm would you choose in general?
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

find the $i^{th}$ smallest element in the array.

What algorithm would you choose in general? Can sorting help?

Algorithm: first sort list and then step through to find $i^{th}$ smallest. What's its runtime?

A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Selection Problem: HOW

Given list of distinct integers \( a_1, a_2, \ldots, a_n \) and integer \( i, 1 \leq i \leq n \), find the \( i^{th} \) smallest element in the array.

What algorithm would you choose in general? Different strategy ...

Pick random list element called “pivot.”

Partition list into those smaller than pivot, those bigger than pivot.

Using \( i \) and size of partition sets, determine in which set to continue looking.
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot. Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$ \hspace{1cm} i = 3
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i, 1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”

Partition list into those smaller than pivot, those bigger than pivot.

Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$  \hspace{1cm} i = 3  \hspace{1cm} \text{Random pivot: 17}
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$  $i = 3$  Random pivot: 17

Smaller than 17: 3, 8, 2  Bigger than 17: 42, 19, 21
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$  \hspace{1cm} i = 3  \hspace{1cm} \text{Random pivot: 17}$

Smaller than 17: 3, 8, 2  \hspace{1cm} Bigger than 17: 42, 19, 21

Has 3 elements so third smallest must be in this set
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$ $i = 3$ Random pivot: 17
   New list: 3, 8, 2 $i = 3$
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i, 1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$  $i = 3$  Random pivot: 17
New list: $3, 8, 2$  $i = 3$  Random pivot: 8
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot. Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$ $i = 3$ Random pivot: 17
New list: $3, 8, 2$ $i = 3$ Random pivot: 8
Smaller than 8: $3, 2$ Bigger than 8: $= 2$
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. \[17, 42, 3, 8, 19, 21, 2\] $i = 3$ Random pivot: 17
New list: 3, 8, 2 $i = 3$ Random pivot: 8

Smaller than 8: 3, 2
Bigger than 8:

Has 2 elements so third smallest must be "next" element, i.e. 8
Selection Problem: HOW

Given list of distinct integers $a_1, a_2, ..., a_n$ and integer $i$, $1 \leq i \leq n$, find the $i^{th}$ smallest element in the array.

Pick random list element called “pivot.”
Partition list into those smaller than pivot, those bigger than pivot.
Using $i$ and size of partition sets, determine in which set to continue looking.

ex. $17, 42, 3, 8, 19, 21, 2$ $i = 3$ Random pivot: 17
New list: $3, 8, 2$ $i = 3$ Random pivot: 8
Smaller than 8: $3, 2$ Bigger than 8:

Return 8 compare to original list: $17, 42, 3, 8, 19, 21, 2$
Selection Problem: HOW

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

Algorithm will incorporate both randomness and recursion!
Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i, 1 \leq i \leq n$, 
$\text{RandSelect}(A, i)$
1. If $n=1$ return $a_1$

What are we doing in this first line?

A. Establishing the base case of the recursion.
B. Establishing the induction step.
C. Randomly picking a pivot.
D. Randomly returning a list element.
E. None of the above.
Selection Problem: HOW

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

$\text{RandSelect}(A,i)$

1. If $n=1$ return $a_1$
2. Initialize lists S and B.
3. Pick integer $j$ uniformly at random from 1 to $n$.
4. For each index $k$ from 1 to $n$ (except $j$):
   5. if $a_k < a_j$, add $a_k$ to the list S.
   6. if $a_k > a_j$, add $a_k$ to the list B.

$$O(1)$$

$$n-1 = O(n)$$
Selection Problem: HOW

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

$\text{RandSelect}(A,i)$

1. If $n=1$ return $a_1$
2. Initialize lists $S$ and $B$.
3. Pick integer $j$ uniformly at random from 1 to $n$.
4. For each index $k$ from 1 to $n$ (except $j$):
5. \hspace{1em} if $a_k < a_j$, add $a_k$ to the list $S$.
6. \hspace{1em} if $a_k > a_j$, add $a_k$ to the list $B$.
7. \hspace{2em} Let $s$ be the size of $S$.
8. \hspace{2em} If $s = i-1$, return $a_j$. 
Selection Problem: HOW

Given list of distinct integers \( A = a_1, a_2, \ldots, a_n \) and integer \( i, 1 \leq i \leq n \),

\[ \text{RandSelect}(A,i) \]

1. If \( n=1 \) return \( a_1 \)
2. Initialize lists \( S \) and \( B \).
3. Pick integer \( j \) uniformly at random from 1 to \( n \).
4. For each index \( k \) from 1 to \( n \) (except \( j \)):
   5. if \( a_k < a_j \), add \( a_k \) to the list \( S \).
   6. if \( a_k > a_j \), add \( a_k \) to the list \( B \).
7. Let \( s \) be the size of \( S \).
8. If \( s = i-1 \), return \( a_j \).
9. If \( s \geq i \), return \( \text{RandSelect}(S, i) \).
10. If \( s < i \), return \( \text{RandSelect}(B, \_???) \).

What's the right way to fill in this blank?

A. \( i \)  
B. \( s \)  
C. \( i+s \)  
D. \( i-(s+1) \)  
E. None of the above.
Selection Problem: WHEN

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i, 1 \leq i \leq n,$

$\text{RandSelect}(A, i)$

1. If $n=1$ return $a_1$
2. Initialize lists $S$ and $B.$
3. Pick integer $j$ uniformly at random from $1$ to $n.$
4. For each index $k$ from $1$ to $n$ (except $j$):
   5. if $a_k < a_j,$ add $a_k$ to the list $S.$
   6. if $a_k > a_j,$ add $a_k$ to the list $B.$
7. Let $s$ be the size of $S.$
8. If $s = i-1,$ return $a_j.$
9. If $s \geq i,$ return $\text{RandSelect}(S, i).$
10. If $s < i,$ return $\text{RandSelect}(B, i-(s+1)).$

What input gives the best-case performance of this algorithm?

A. When element we're looking for is the first in list.
B. When element we're looking for is $i^{th}$ in list.
C. When element we're looking for is in the middle of the list.
D. When element we're looking for is last in list.
E. None of the above.
Selection Problem: WHEN

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$, $RandSelect(A, i)$

1. If $n=1$ return $a_1$
2. Initialize lists $S$ and $B$.
3. Pick integer $j$ uniformly at random from 1 to $n$.
4. For each index $k$ from 1 to $n$ (except $j$):
   5. if $a_k < a_j$, add $a_k$ to the list $S$.
   6. if $a_k > a_j$, add $a_k$ to the list $B$.
7. Let $s$ be the size of $S$.
8. If $s = i-1$, return $a_j$.
9. If $s \geq i$, return $RandSelect(S, i)$.
10. If $s < i$, return $RandSelect(B, i-(s+1))$. 

Performance depends on more than the input!
Selection Problem: WHEN

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$ and integer $i$, $1 \leq i \leq n$,

$\text{RandSelect}(A, i)$

1. If $n=1$ return $a_1$
2. Initialize lists $S$ and $B$.
3. **Pick integer $j$ uniformly at random from 1 to $n$.**
4. For each index $k$ from 1 to $n$ (except $j$):
   5. if $a_k < a_j$, add $a_k$ to the list $S$.
   6. if $a_k > a_j$, add $a_k$ to the list $B$.
7. Let $s$ be the size of $S$.
8. **If $s = i-1$, return $a_j$.**
9. If $s \geq i$, return $\text{RandSelect}(S, i)$.
10. If $s < i$, return $\text{RandSelect}(B, i-(s+1))$.

Minimum time if we happen to pick pivot which is the $i^{th}$ smallest list element.

In this case, what's the runtime?

A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
The runtime of Selection not only depends on the size of the input but it also depends on the random choices.
If you randomly select the $i$th element, then your list will be split into a list of length $i$ and a list of length $n-i$.

So when we recurse on the smaller lists, it will take time proportional to $\max(i, n - i)$.
Expected runtime

Clearly, the split with the smallest maximum size is when $i=n/2$

and worst case is $i=n$ or $i=1$. 
What is the expected runtime?

Well what is our random variable?

For each input and sequence of random choices of pivots, the random variable is the runtime of that particular outcome.
So if we want to find the expected runtime, we must sum over all possibilities of choices.

Let \( ET(n) \) be the expected runtime. Then

\[
ET(n) = \sum_{i=1}^{n} ET(\max(i, n - i)) + O(n)
\]
What is the probability of choosing a value from 1 to $n$ in the interval $\left[\frac{n}{4}, \frac{3n}{4}\right]$ if all values are equally likely?

A: $n/2$
B: $\frac{1}{4}$
C: $\frac{1}{2}$
D: $3/2$
E: depends on $n$
If you did choose a value between $n/4$ and $3n/4$ then the sizes of the subproblems would both be $\leq \frac{3n}{4}$.

Otherwise, the subproblems would be $\leq n$.

So we can compute an upper bound on the expected runtime.

\[
ET(n) \leq \frac{1}{2} ET\left(\frac{3n}{4}\right) + \frac{1}{2} ET(n) + O(n)
\]
Expected runtime

\[ T(n) = aT\left(\frac{n}{b}\right) + O\left(n^d\right) \]

\[ ET(n) \leq \frac{1}{2} ET\left(\frac{3n}{4}\right) + \frac{1}{2} ET(n) + O(n) \]

\[ \frac{1}{2} ET(n) \leq \frac{1}{2} ET\left(\frac{3n}{4}\right) + O(n) \]

Plug into the master theorem with \( a=1, b=4/3, d=1 \)

\( a < b^d \) so

\[ ET(n) \leq O(n) \]

But we can argue that this is also a lower bound so….

\[ ET(n) = O(n) \]
Selection Problem: WHEN

**Situation so far:**

Sort then search takes worst-case \( \Theta(n \log n) \)

Randomized selection takes worst-case expected time \( \Theta(n) \)

Unlucky \( n + n-1 + n-2 + \ldots = \binom{n-1}{2} \)
Selection Problem: WHEN

**Situation so far:**

Sort then search takes worst-case $\Theta(n \log n)$

Randomized selection takes worst-case expected time $\Theta(n)$

*How do we implement randomized algorithms?*

*Are there deterministic algorithms that perform as well?*

For selection problem: Blum et al, yes!

In general: open 😊
Element Distinctness: WHAT

Given list of positive integers $a_1, a_2, \ldots, a_n$ decide whether all the numbers are distinct or whether there is a repetition, i.e. two positions $i, j$ with $1 \leq i < j \leq n$ such that $a_i = a_j$.

What algorithm would you choose in general?
Element Distinctness: HOW

Given list of positive integers $a_1, a_2, \ldots, a_n$ decide whether all the numbers are distinct or whether there is a repetition, i.e. two positions $i, j$ with $1 \leq i < j \leq n$ such that $a_i = a_j$.

*What algorithm would you choose in general? Can sorting help?*

Algorithm: first sort list and then step through to find duplicates. What's its runtime?

A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Given list of positive integers $a_1, a_2, \ldots, a_n$ decide whether all the numbers are distinct or whether there is a repetition, i.e. two positions $i, j$ with $1 \leq i < j \leq n$ such that $a_i = a_j$.

What algorithm would you choose in general? Can sorting help?

Algorithm: first sort list and then step through to find duplicates. How much memory does it require?
A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Element Distinctness: HOW

Given list of positive integers $a_1, a_2, \ldots, a_n$ decide whether all the numbers are distinct or whether there is a repetition, i.e. two positions $i, j$ with $1 \leq i < j \leq n$ such that $a_i = a_j$.

What algorithm would you choose in general? What if we had unlimited memory?
Given list of positive integers $A = a_1, a_2, \ldots, a_n$,

**UnlimitedMemoryDistinctness**($A$)
1. For $i = 1$ to $n$, 
2. If $M[a_i] = 1$ then return "Found repeat"
3. Else $M[a_i] := 1$
4. Return "Distinct elements"

What's the runtime of this algorithm?
A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Given list of positive integers $A = a_1, a_2, ..., a_n$,

UnlimitedMemoryDistinctness($A$)
1. For $i = 1$ to $n$,
2. If $M[a_i] = 1$ then return "Found repeat"
3. Else $M[a_i] := 1$
4. Return "Distinct elements"

M is an array of memory locations
This is memory location indexed by $a_i$
To simulate having more memory locations: use **Virtual Memory**.

Define **hash function**

\[ h: \{ \text{desired memory locations} \} \rightarrow \{ \text{actual memory locations} \} \]

- Typically we want more memory than we have, so \( h \) is **not one-to-one**.
- How to implement \( h \)?
  - CSE 12, CSE 100.
- Here, let's use hash functions in an algorithm for Element Distinctness.
Element Distinctness: HOW

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

$\text{HashDistinctness}(A, m)$
1. Initialize array $M[1,..,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to $1,\ldots,m$.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"
Element Distinctness: HOW

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

**HashDistinctness($A$, $m$)**
1. Initialize array $M[1,..,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"

What's the runtime of this algorithm?
A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Element Distinctness: HOW

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

HashDistinctness($A$, $m$)
1. Initialize array $M[1,..,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to 1,..,$m$.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"

What's the memory use of this algorithm?
A. $\Theta(1)$
B. $\Theta(n)$
C. $\Theta(n \log n)$
D. $\Theta(n^2)$
E. None of the above
Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

$\text{HashDistinctness}(A, m)$
1. Initialize array $M[1,\ldots,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to 1,\ldots,m.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"

But this algorithm might make a mistake!!!
When?
Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available.

**HashDistinctness($A$, $m$)**
1. Initialize array $M[1,..,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"

**Correctness:** _Goal is_
If there is a repetition, algorithm finds it
If there is no repetition, algorithm reports "Distinct elements"
Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

**HashDistinctness($A$, $m$)**
1. Initialize array $M[1,..,m]$ to all 0s.
2. Pick a hash function $h$ from all positive integers to 1,..,$m$.
3. For $i = 1$ to $n$,
4. If $M[h(a_i)] = 1$ then return "Found repeat"
5. Else $M[h(a_i)] := 1$
6. Return "Distinct elements"

**Correctness:** Goal is
If there is a repetition, algorithm finds it
If there is no repetition, algorithm reports "Distinct elements"
Resolving collisions with chaining

Hash Table

Each memory location holds a pointer to a linked list, initially empty.

Each linked list records the items that map to that memory location.

Collision means there is more than one item in this linked list.
Element Distinctness: HOW

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

\textbf{ChainHashDistinctness}(A, m)
1. Initialize array $M[1,\ldots,m]$ to null lists.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For $i = 1$ to $n$,
4. For each element $j$ in $M[h(a_i)]$,
5. If $a_j = a_i$ then return "Found repeat"
6. Append $i$ to the tail of the list $M[h(a_i)]$
7. Return "Distinct elements"
Element Distinctness: WHY

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

\[\text{ChainHashDistinctness}(A, m)\]
1. Initialize array $M[1,..,m]$ to null lists.
2. Pick a hash function $h$ from all positive integers to $1,..,m$.
3. For $i = 1$ to $n$,
4. For each element $j$ in $M[h(a_i)]$,
5. If $a_j = a_i$ then return "Found repeat"
6. Append $i$ to the tail of the list $M[h(a_i)]$
7. Return "Distinct elements"

**Correctness:** *Goal is*
If there is a repetition, algorithm finds it ✓
If there is no repetition, algorithm reports "Distinct elements" ✓
Element Distinctness: MEMORY

Given list of positive integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

**ChainHashDistinctness**(A, m)
1. Initialize array $M[1,..,m]$ to null lists.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For $i = 1$ to $n$,
4. For each element $j$ in $M[ h(a_i) ]$,
5. If $a_j = a_i$ then return "Found repeat"
6. Append $i$ to the tail of the list $M[ h(a_i) ]$
7. Return "Distinct elements"

What's the memory use of this algorithm?
Element Distinctness: MEMORY

Given list of distinct integers $A = a_1, a_2, \ldots, a_n$, and $m$ memory locations available

**ChainHashDistinctness($A$, $m$)**
1. Initialize array $M[1,\ldots,m]$ to null lists.
2. Pick a hash function $h$ from all positive integers to $1,\ldots,m$.
3. For $i = 1$ to $n$,
4. For each element $j$ in $M[ h(a_i) ]$,
5. If $a_j = a_i$ then return "Found repeat"
6. Append $i$ to the tail of the list $M[ h(a_i) ]$
7. Return "Distinct elements"

**What's the memory use of this algorithm?**
Size of $M$: $O(m)$. Total size of all the linked lists: $O(n)$. Total memory: $O(m+n)$. 
Element Distinctness: WHEN

\[ \text{ChainHashDistinctness}(A, m) \]
1. Initialize array \( M[1,\ldots,m] \) to null lists. \( \Theta(1) \)
2. Pick a hash function \( h \) from all positive integers to 1,\ldots,m. \( \Theta(1) \)
3. For \( i = 1 \) to \( n \),
   4. For each element \( j \) in \( M[ h(a_i) ] \),
5. If \( a_j = a_i \) then return "Found repeat"
6. Append \( i \) to the tail of the list \( M[ h(a_i) ] \)
7. Return "Distinct elements" \( \Theta(1) \)
Element Distinctness: WHEN

ChainHashDistinctness(A, m)
1. Initialize array M[1,..,m] to null lists.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For i = 1 to n,
   4. For each element j in M[ $h(a_i)$ ],
      5. If $a_j = a_i$ then return "Found repeat"
   6. Append i to the tail of the list M [ $h(a_i)$ ]
7. Return "Distinct elements"

Worst case is when we don't find $a_i$: $O(1 + \text{size of list } M[ h(a_i) ])$
ChainHashDistinctness(A, m)

1. Initialize array M[1,..,m] to null lists.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For i = 1 to n,
   4. For each element $j$ in M[ $h(a_i)$ ],
   5. If $a_j = a_i$ then return "Found repeat"
   6. Append i to the tail of the list M[ $h(a_i)$ ]
7. Return "Distinct elements"

Worst case is when we don't find $a_i$:

$$O(1 + \text{size of list } M[ h(a_i) ])$$

$$= O(1 + \# j<i \text{ with } h(a_j)=h(a_i))$$
Element Distinctness: WHEN

ChainHashDistinctness(A, m)
1. Initialize array M[1,..,m] to null lists.
2. Pick a hash function $h$ from all positive integers to 1,..,m.
3. For $i = 1$ to $n$,
4. For each element $j$ in $M[ h(a_i) ]$,
5. If $a_j = a_i$ then return "Found repeat"
6. Append $i$ to the tail of the list $M[ h(a_i) ]$
7. Return "Distinct elements"

Worst case is when we don't find $a_i$: O( $1 + \text{size of list } M[ h(a_i) ]$ )
= O( $1 + \# j<i \text{ with } h(a_j)=h(a_i)$ )

Total time: $O(n + \sum_{i=1}^{n} \# \text{ collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i )$
= $O(n + \text{total # collisions})$
Element Distinctness: WHEN

Collisions depend on choice of **hash function**

\[ h: \{ \text{desired memory locations} \} \rightarrow \{ \text{actual memory locations} \} \]

**Ideal hash function model**: each output in \( \{1,2,\ldots,m\} \) is equally likely.

So \( h \) is a function that chooses a random number in \( \{1,2,\ldots,m\} \) for each input \( a_i \).
Element Distinctness: WHEN

Total time: \( O(n + \sum_{i=1}^{n} \# \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i ) \)

\[= O(n + \text{total # collisions}) \]

What's the expected total number of collisions?
Element Distinctness: WHEN

**Total time:** $O(n + \sum_{i=1}^{n} \text{# collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i)$

$$= O(n + \text{total # collisions})$$

**What's the expected total number of collisions?**

For each pair $(i,j)$ with $j<i$, define:

$$X_{i,j} = 1 \text{ if } h(a_i)=h(a_j) \text{ and } X_{i,j}=0 \text{ otherwise.}$$

**Total # of collisions =** $\sum_{(i,j):j<i} X_{i,j}$
Element Distinctness: WHEN

Total time: $O(n + \sum_{i=1}^{n} \# \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i) = O(n + \text{total # collisions})$

What's the expected total number of collisions?

For each pair $(i,j)$ with $j<i$, define:

$$X_{i,j} = 1 \text{ if } h(a_i) = h(a_j) \text{ and } X_{i,j} = 0 \text{ otherwise.}$$

$$\text{Total # of collisions} = \sum_{(i,j): j<i} X_{i,j}$$

So by linearity of expectation: $E(\text{total # of collisions}) = \sum_{(i,j): j<i} E(X_{i,j})$
Element Distinctness: WHEN

**Total time**: $O(n + \sum_{i=1}^{n} \# \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i)$

$= O(n + \text{total # collisions})$

What's the expected total number of collisions?

For each pair $(i,j)$ with $j<i$, define:

$X_{i,j} = 1$ if $h(a_i)=h(a_j)$ and $X_{i,j}=0$ otherwise.

Total # of collisions $= \sum_{(i,j): j<i} X_{i,j}$

What's $E(X_{i,j})$?

A. $1/n$
B. $1/m$
C. $1/n^2$
D. $1/m^2$
E. None of the above.
Element Distinctness: WHEN

**Total time:** \( O(n + \sum_{i=1}^{n} \# \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i ) \)

\[ \text{Total number of collisions} = O(n + \text{total } \# \text{collisions}) \]

*What's the expected total number of collisions?*

For each pair \((i,j)\) with \(j<i\), define:

\[ X_{i,j} = 1 \text{ if } h(a_i) = h(a_j) \text{ and } X_{i,j} = 0 \text{ otherwise.} \]

**Total # of collisions = \( \sum_{(i,j): j<i} X_{i,j} \)**

How many terms are in the sum? That is, how many pairs \((i,j)\) with \(j<i\) are there?

A. \(n\)
B. \(n^2\)
C. \(C(n,2)\)
D. \(n(n-1)\)
Element Distinctness: WHEN

**Total time:** \( O(n + \sum_{i=1}^{n} # \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i ) ) \)

\[ = O(n + \text{total # collisions}) \]

**What's the expected total number of collisions?**

For each pair \((i,j)\) with \(j<i\), define:

\[ X_{i,j} = 1 \text{ if } h(a_i) = h(a_j) \text{ and } X_{i,j} = 0 \text{ otherwise.} \]

So by linearity of expectation:

\[
E(\text{total # of collisions}) = \sum_{(i,j):j<i} E(X_{i,j}) = \binom{n}{2} \frac{1}{m} = O(n^2/m)
\]
Element Distinctness: WHEN

Total time: \( O(n + \sum_{i=1}^{n} \# \text{collisions between pairs } a_i \text{ and } a_j, \text{ where } j<i ) \)

= \( O(n + \text{total # collisions}) \)

Total expected time: \( O(n + n^2/m) \)

In ideal hash model, as long as \( m>n \) the total expected time is \( O(n) \).
HW 8 due **Wednesday** at 11:59pm via Gradescope.

**Final exam:**

<table>
<thead>
<tr>
<th>Section</th>
<th>Date</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A00</td>
<td>Wed, March 16</td>
<td>8:00am - 11:00am</td>
</tr>
<tr>
<td>B00</td>
<td>Mon, March 14</td>
<td>3:00pm - 6:00pm</td>
</tr>
<tr>
<td>C00</td>
<td>Mon, March 14</td>
<td>11:30am - 2:30pm</td>
</tr>
</tbody>
</table>

See website for practice final, review session details, seating charts. Review sessions are Thursday evening and Saturday at noon.