Encoding/Decoding, Lower Bounds

Russell Impagliazzo and Miles Jones
Thanks to Janine Tiefenbruck

http://cseweb.ucsd.edu/classes/sp16/cse21-bd/
May 9, 2016
Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example: $n=12$, $k=3$, window size $n/k = 4$.

How do we encode $s = 011000000010$?

Output:

$$
\left( \frac{12}{3} \right) = \frac{2 \cdot 11 \cdot 10}{32 \cdot 1} = 220
$$

You need minimum 8 bits.
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example $n=12$, $k=3$, window size $n/k = 4$.

How do we encode $s = 011000000010$? Output: 101

Interpret next bits as position of 1; this position is 01
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode $s = 011000000010$ ? Output: 101 150
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 01\underline{100000010} ? Output: 101100

Interpret next bits as position of 1; this position is 00
**Encoding: Fixed Density Strings**

**Idea:** use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

**Example** \( n=12, k=3, \text{window size } n/k = 4 \).

How do we encode \( s = 011000000010 \)?  

Output: 101100
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example $n=12$, $k=3$, window size $n/k = 4$.

How do we encode $s = 011\underline{000}00010$ ?

Output: $1011000$

No 1s in this window.
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 101100011
Idea: use marker bit to indicate when to interpret output as a position.
    - Fix window size.
    - If there is a 1 in the current "window" in the string, 
      record a 1 to interpret next bits as position, 
      then record its position and move the window over. 
    - Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 01100000000010 ? Output: 1011000111

Interpret next bits as position of 1; this position is 11
Idea: use marker bit to indicate when to interpret output as a position.
- Fix window size.
- If there is a 1 in the current "window" in the string, record a 1 to interpret next bits as position, then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 1011000111

Now we can stop recording, since we have seen all three ones.
procedure WindowEncode (input: $b_1b_2...b_n$, with exactly $k$ ones and $n-k$ zeros)

1. $w := \text{floor} \ (n/k)$
2. count := 0
3. location := 1
4. While count < $k$:
   5. If there is a 1 in the window starting at current location
      6. Output 1 as a marker, then output position of first 1 in window.
      7. Increment count.
      8. Update location to immediately after first 1 in this window.
   9. Else
      10. Output 0.
      11. Update location to next index after current window.

Uniquely decodable?
Decoding: Fixed Density Strings

procedure WindowDecode (input: \( x_1 x_2 \ldots x_m \), target is exactly \( k \) ones and \( n-k \) zeros)

1. \( w := \text{floor} \left( \frac{n}{k} \right) \)
2. \( b := \text{floor} \left( \log_2(w) \right) \)
3. \( s := \text{empty string} \)
4. \( i := 0 \)
5. While \( i < m \)
6. \hspace{1em} If \( x_i = 0 \)
7. \hspace{2em} \( s += 0 \ldots 0 \) (\( w \) times)
8. \hspace{2em} \( i += 1 \)
9. \hspace{1em} Else
10. \hspace{2em} \( p := \text{decimal value of the bits} \ x_{i+1} \ldots x_{i+b} \)
11. \hspace{2em} \( s += 0 \ldots 0 \) (\( p \) times)
12. \hspace{2em} \( s += 1 \)
13. \hspace{2em} \( i := i+b+1 \)
14. If length\( (s) < n \)
15. \hspace{2em} \( s += 0 \ldots 0 \) (\( n-\text{length}(s) \) times )
16. Output \( s \).
Encoding/Decoding: Fixed Density Strings

Correctness?

$E(s) =$ result of encoding string $s$ of length $n$ with $k$ 1s, using $\text{WindowEncode}$.

$D(t) =$ result of decoding string $t$ to create a string of length $n$ with $k$ 1s, using $\text{WindowDecode}$.

Well-defined functions?

Inverses?

Goal: For each $s$, $D(E(s)) = s$.

Strong Induction!
Output size?

In our example: \( n = 12 \), \( k = 3 \). \( s = 101 \ldots \).

How many bits was \( E(s) \)?: \( 10 \)

Assume \( n/k \) is a power of two. Consider \( s \) a binary string of length \( n \) with \( k \) 1s.

How many bits is \( E(s) \)?

A. \( n-1 \)
B. \( \log_2(n/k) \)
C. Depends on where 1s are located in \( s \)
Assume $n/k$ is a power of two. Consider $s$ a binary string of length $n$ with $k$ 1s.

For which strings is $E(s)$ shortest?

A. More 1s toward the beginning.
B. More 1s toward the end.
C. 1s spread evenly throughout.
Output size?

Assume $n/k$ is a power of two. Consider $s$ a binary string of length $n$ with $k$ 1s.

**Best case**: 1s toward the beginning of the string. $E(s)$ has
- One bit for each 1 in $s$ to indicate that next bits denote positions in window.
- $\log_2(n/k)$ bits for each 1 in $s$ to specify position of that 1 in a window.
- $k$ such 1s.
- No bits representing 0s because all 0s occur in windows with 1s or after the last 1.

Total $|E(s)| = k \log_2(n/k) + k$
Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

Worst case: 1s toward the end of the string. E(s) has
- Some bits representing 0s since there are no 1s in first several windows.
- One bit for each 1 in s to indicate that next bits denote positions in window.
- \( \log_2(n/k) \) bits for each 1 in s to specify position of that 1 in a window.
- k such 1s.

What's an upper bound on the number of these bits?
A. n  
B. n-k  
C. k  
D. 1  
E. None of the above.
Output size?

Assume \( n/k \) is a power of two. Consider \( s \) a binary string of length \( n \) with \( k \) 1s.

**Worst case**: 1s toward the end of the string. \( E(s) \) has
- At most \( k \) bits representing 0s since there are no 1s in first several windows.
- One bit for each 1 in \( s \) to indicate that next bits denote positions in window.
- \( \log_2(n/k) \) bits for each 1 in \( s \) to specify position of that 1 in a window.
- \( k \) such ones.

**Total** \( |E(s)| \leq k \log_2(n/k) + 2k \)
Output size?

Assume $n/k$ is a power of two. Consider $s$ a binary string of length $n$ with $k$ 1s.

$$k \log_2(n/k) + k \leq |E(s)| \leq k \log_2(n/k) + 2k$$

number of $1,4,3$
Output size?

Assume \( n/k \) is a power of two. Consider \( s \) a binary string of length \( n \) with \( k \) 1s. Given \(|E(s)| \leq k \log_2(n/k) + 2k\), we need at most \( k \log_2(n/k) + 2k \) bits to represent all length \( n \) binary strings with \( k \) 1s. Hence, there are at most \( 2^{\cdot\cdot\cdot} \) many such strings.
Output size?

Assume $n/k$ is a power of two. Consider $s$ a binary string of length $n$ with $k$ 1s. Given $|E(s)| \leq k \log_2(n/k) + 2k$, we need at most $k \log_2(n/k) + 2k$ bits to represent all length $n$ binary strings with $k$ 1s. Hence, there are at most $2^{\cdots}$ many such strings.

$$2^{(k \log(n/k) + 2k)} = 2^{(k \log(n/k))} \cdot 2^{(2k)}$$

$$= \left(2^{(\log(n/k))}\right)^{k} \cdot 2^{(2k)}$$

$$= (n/k)^k \cdot 4^k = (4n/k)^k$$

$$\binom{n}{k} \cdot \left(\frac{4n}{k}\right)^k = \tilde{O}(n^k)$$
Output size?

Assume \( n/k \) is a power of two. Consider \( s \) a binary string of length \( n \) with \( k \) 1s. Given \( |E(s)| \leq k \log_2(n/k) + 2k \), we need at most \( k \log_2(n/k) + 2k \) bits to represent all length \( n \) binary strings with \( k \) 1s. Hence, there are at most \( 2^{(k \log_2(n/k) + 2k)} \) many such strings.

\[
2^{(k \log_2(n/k) + 2k)} = 2^{(k \log_2(n/k))} \cdot 2^{(2k)}
\]

\[
= \left(2^{(\log_2(n/k))}\right)^k \cdot 2^{(2k)}
\]

\[
= \left(\frac{n}{k}\right)^k \cdot 4^k = (4n/k)^k
\]

\[
C(n,k) = \# \text{ Length } n \text{ binary strings with } k \text{ 1s} \leq (4n/k)^k
\]
Using $\text{windowEncode}()$: $\binom{n}{k} \leq (4n/k)^k$

Lower bound?

Idea: find a way to count a subset of the fixed density binary strings.

Some fixed density binary strings have one 1 in each of $k$ chunks of size $n/k$.

How many such strings are there?

A. $n^n$    B. $k!$    C. $(n/k)^k$    D. $C(n,k)^k$    E. None of the above.
Bounds for Binomial Coefficients

Using `windowEncode()`:

\[ \binom{n}{k} \leq (4n/k)^k \]

Using evenly spread strings:

\[ (n/k)^k \leq \binom{n}{k} \leq \left( \frac{4n}{k} \right)^k \]

\[ \binom{n}{k} = \Theta \left( n^k \right) \]

**Counting** helps us analyze our compression algorithm.

**Compression algorithms** help us count.
A **theoretically optimal encoding** for length $n$ binary strings with $k$ 1s would use the ceiling of $\log_2 \binom{n}{k}$ bits.

**How?**
- List all length $n$ binary strings with $k$ 1s in some order.
- **To encode:** Store the **position** of a string in the list, rather than the string itself.
- **To decode:** Given a position in list, need to determine string in that position.
A theoretically optimal encoding for length $n$ binary strings with $k$ 1s would use the ceiling of $\log_2 \binom{n}{k}$ bits.

**How?**
- List all length $n$ binary strings with $k$ 1s in some order.
- To encode: Store the position of a string in the list, rather than the string itself.
- To decode: Given a position in list, need to determine string in that position.

Use lexicographic (dictionary) ordering …
Lex Order

String $a$ comes before string $b$ if the first time they differ, $a$ is smaller.

i.e.

$$a_1a_2...a_n <_{\text{lex}} b_1b_2...b_n$$

means there exists $j$ such that

$$a_i=b_i \text{ for all } i<j \text{ AND } a_j<b_j$$

Which of these comes last in lex order?

A. 1001
B. 0011
C. 1101
D. 1010
E. 0000
E.g. Length n=5 binary strings with k=3 ones, listed in lex order:

<table>
<thead>
<tr>
<th>Original string, s</th>
<th>Encoded string (i.e. position in this list)</th>
</tr>
</thead>
<tbody>
<tr>
<td>00111</td>
<td>0 = 0000</td>
</tr>
<tr>
<td>01011</td>
<td>1 = 0001</td>
</tr>
<tr>
<td>01101</td>
<td>2 = 0010</td>
</tr>
<tr>
<td>01110</td>
<td>3 = 0011</td>
</tr>
<tr>
<td>10011</td>
<td>4 = 0100</td>
</tr>
<tr>
<td>10101</td>
<td>5 = 0101</td>
</tr>
<tr>
<td>10110</td>
<td>6 = 0110</td>
</tr>
<tr>
<td>11001</td>
<td>7 = 0111</td>
</tr>
<tr>
<td>11010</td>
<td>8 = 1000</td>
</tr>
<tr>
<td>11100</td>
<td>9 = 1001</td>
</tr>
</tbody>
</table>
Need two algorithms, given specific $n$ and $k$:

\[
s \rightarrow E(s, n, k)
\]

and

\[
p \rightarrow D(p, n, k)
\]

**Idea:** Use recursion (reduce & conquer).
Lex Order: Algorithm?

For \( E(s,n,k) \):

- Any string that starts with 0 must have position \( \text{before} \) \( \binom{n-1}{k} \)
- Any string that starts with 1 must have position \( \text{at or after} \) \( \binom{n-1}{k} \)

Length \( n-1 \) binary strings with \( k \) 1s

Length \( n-1 \) binary strings with \( k-1 \) 1s
Lex Order: Algorithm?

For $E(s,n,k)$:

- Any string that starts with 0 must have position before $\binom{n-1}{k}$.
- Any string that starts with 1 must have position at or after $\binom{n-1}{k}$.

procedure lexEncode ($b_1b_2...b_n$, n, k)

1. If $n = 1$,
2. return 0.
3. If $s_1 = 0$,
4. return lexEncode ($b_2...b_n$, n-1, k)
5. Else
6. return $C(n-1,k) + \text{lexEncode}(b_2...b_n, n-1, k-1)$
For \( D(s,n,k) \):

- Any position before \( \binom{n-1}{k} \) must correspond to string that starts with 0.
- Any position at or after \( \binom{n-1}{k} \) must correspond to string that starts with 1.

**procedure lexDecode \((p, n, k)\)**

1. If \( n = k \),
2. return 1111..1  //length n string of all 1s
3. If \( p < C(n-1,k) \),
4. return "0" + lexDecode\((p, n-1, k)\)
5. Else
6. return "1" + lexDecode\((p-C(n-1,k), n-1, k-1)\)
Victory!

Using lexEncode, lexDecode, we can represent any fixed density length n binary string with k 1s as a number in the range 0 through \( C(n,k) - 1 \).

So, it takes \( \log_2(C(n,k)) \) bits to store fixed-density binary strings using lex order.

**Theoretical lower bound**: \( \log_2(C(n,k)) \).

**Same!** So this encoding algorithm is optimal.

*From a \( n \)-bit string down to a \( \log_2(C(n,k)) \)-bit string.*
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements

*What's the *fastest possible worst case* for any sorting algorithm?*
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements

*What's the fastest possible worst case* for any sorting algorithm?

**Tree diagram** represents possible comparisons we might have to do, based on relative sizes of elements.
Another application of counting ... lower bounds

Tree diagram represents possible comparisons we might have to do, based on relative sizes of elements.

Rosen p. 761
Another application of counting ... lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements.

What's the fastest possible worst case for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.
How many leaves will there be in a decision tree that sorts n elements?

A. $2^n$
B. $\log n$
C. $n!$
D. $C(n,2)$
E. None of the above.
Another application of counting ... lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements.

What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?

* Internal nodes correspond to comparisons.
* Leaves correspond to possible input arrangements.
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements

What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?
- * Internal nodes correspond to comparisons. * Leaves correspond to possible input arrangements.

  - Depends on algorithm.
  - n!
Another application of counting ... lower bounds

How does height relate to number of leaves?

**Theorem**: There are at most $2^h$ leaves in a binary tree with height $h$.

Which of the following is true?

A. It's possible to have a binary tree with height 3 and 1 leaf.
B. It's possible to have a binary tree with height 1 and 3 leaves.
C. Every binary tree with height 3 has 1 leaf.
D. Every binary tree with height 1 has 3 leaves.
E. None of the above.
Another application of counting ... lower bounds

How does height relate to number of leaves?

**Theorem**: There are at most \(2^h\) leaves in a binary tree with height \(h\).

**Proof**: By *induction* on the height \(h \geq 0\).

*Base case* WTS that there are at most 2^0 leaves in a binary tree with height \(h=0\).

*What trees have height 0?*
How does height relate to number of leaves?

**Theorem:** There are at most $2^h$ leaves in a binary tree with height $h$.

**Proof:** By induction on the height $h \geq 0$.

**Base case** WTS that there are at most $2^0$ leaves in a binary tree with height $h=0$.

If a binary tree has height 0, its only node is the root. In this case the root is also a (and the only) leaf node. So, the number of leaves is $1 = 2^0$ in the only possible tree with $h=0$. 😊
Another application of counting ... lower bounds

How does height relate to number of leaves?

**Theorem:** There are at most $2^h$ leaves in a binary tree with height $h$.

**Proof:** By *induction* on the height $h \geq 0$.

*Induction Step* Let $h$ be some integer $\geq 0$ and assume (as the **IH**) that

There are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$. 
**Induction Step** Let $h$ be some integer $\geq 0$ and assume (as the IH) that there are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$. Consider a tree $U$ with height $h+1$. How can we relate it to trees of height $h$ so that we can use IH?
Another application of counting … lower bounds

*Induction Step* Let \( h \) be some integer \( \geq 0 \) and assume (as the IH) that

There are at most \( 2^h \) leaves in a binary tree with height \( h \).

WTS that there are at most \( 2^{h+1} \) leaves in a binary tree with height \( h+1 \).

Consider a tree \( U \) with height \( h+1 \). *How can we relate it to trees of height \( h \) so that we can use IH?*

Remove all the leaves of \( U \). This gives a new tree, \( T \), of height \( h \).

By the IH the tree \( T \) has at most \( 2^h \) leaves.

To get from \( T \) to \( U \), we need to add back the leaves of \( U \). *How many are there?*
Another application of counting ... lower bounds

*Induction Step* Let $h$ be some integer $\geq 0$ and assume (as the **IH**) that

There are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$.
Consider a tree $U$ with height $h+1$. *How can we relate it to trees of height $h$ so that we can use IH?*

**Remove** all the leaves of $U$. This gives a new tree, $T$, of height $h$.
By the **IH** the tree $T$ has at most $2^h$ leaves.

To get from $T$ to $U$, we need to add back the leaves of $U$. *How many are there?*
For each leaf of $T$, there are **at most** 2 leaves in $U$. 
Another application of counting … lower bounds

**Induction Step** Let \( h \) be some integer \( \geq 0 \) and assume (as the IH) that

There are at most \( 2^h \) leaves in a binary tree with height \( h \).

WTS that there are at most \( 2^{h+1} \) leaves in a binary tree with height \( h+1 \). Consider a tree \( U \) with height \( h+1 \). How can we relate it to trees of height \( h \) so that we can use IH?

**Remove** all the leaves of \( U \). This gives a new tree, \( T \), of height \( h \). By the IH the tree \( T \) has at most \( 2^h \) leaves.

To get from \( T \) to \( U \), we need to add back the leaves of \( U \). How many are there? For each leaf of \( T \), there are at most 2 leaves in \( U \).

\[
\text{# leaves in } U \leq 2(\text{# leaves in } T) \leq 2(2^h) = 2^{h+1}
\]
Another application of counting … lower bounds

What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?
  * Internal nodes correspond to comparisons.  
  * Leaves correspond to possible input arrangements.  

Each tree diagram must have at least **n! leaves**, so its height must be at least \( \log_2(n!) \).
What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?

* Internal nodes correspond to comparisons.  
  
  Depends on algorithm.

* Leaves correspond to possible input arrangements.  
  
  \( n! \)

Each tree diagram must have at least **\( n! \) leaves**, so its height must be at least \( \log_2(n!) \).

i.e. fastest possible worst case performance of sorting is \( \log_2(n!) \)
Another application of counting … lower bounds

What's the **fastest possible worst case** for any sorting algorithm? \( \log_2(n!) \)

How big is that?

**Lemma:** For \( n > 1 \),

\[
\left( \frac{n}{2} \right)^{\frac{n}{2}} < n! < n^n
\]

**Proof:**

\[
n! = (n)(n-1)(n-2)\ldots\left(\frac{n}{2}\right)\ldots(3)(2)(1)
\]

\[
> \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\ldots\left(\frac{n}{2}\right)
\]

\[
= \left(\frac{n}{2}\right)^{\frac{n}{2}}
\]

\[
n! = (n)(n-1)(n-2)\ldots(3)(2)(1)
\]

\[
< (n)(n)(n)\ldots(n)(n)(n)
\]

\[
= n^n
\]
Another application of counting … lower bounds

What's the **fastest possible worst case** for any sorting algorithm? \( \log_2(n!) \)

How big is that?

**Lemma:** for \( n>1 \), \( \left( \frac{n}{2} \right)^{\frac{n}{2}} < n! < n^n \)

**Theorem:** \( \log_2(n!) \) is in \( \Theta(n \log n) \)

**Proof:** For \( n>1 \), taking logarithm of both sides in lemma gives

\[ \frac{n}{2} \log \left( \frac{n}{2} \right) < \log_2(n!) < n \log n \]

i.e.

\[ \frac{1}{2} (n \log n - n \log 2) < \log_2(n!) < n \log n \]
Another application of counting ... lower bounds

What's the fastest possible worst case for any sorting algorithm? $\log_2(n!)$

How big is that?

**Lemma:** for $n>1$, $\left(\frac{n}{2}\right)^{\frac{n}{2}} < n! < n^n$

**Theorem:** $\log_2(n!)$ is in $\Theta(n \log n)$

*Therefore,*

the best sorting algorithms will need $\Theta(n \log n)$ comparisons in the worst case.

i.e. it's impossible to have a comparison-based algorithm that does better than **Merge Sort** (in the worst case).
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements

What's the *fastest possible worst case* for any sorting algorithm?
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements.

*What's the *fastest possible worst case* for any sorting algorithm?*

**Tree diagram** represents possible comparisons we might have to do, based on relative sizes of elements.
Another application of counting … lower bounds

**Tree diagram** represents possible comparisons we might have to do, based on relative sizes of elements.

Rosen p. 761
Another application of counting ... lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements.

What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.
How many leaves will there be in a decision tree that sorts $n$ elements?

A. $2^n$
B. $\log n$
C. $n!$
D. $C(n,2)$
E. None of the above.
Sorting algorithm: performance was measured in terms of number of comparisons between list elements.

What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?
- Internal nodes correspond to comparisons.
- Leaves correspond to possible input arrangements.
Another application of counting … lower bounds

**Sorting algorithm:** performance was measured in terms of number of comparisons between list elements.

What's the fastest possible worst case for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?

* Internal nodes correspond to comparisons. * Depends on algorithm.
* Leaves correspond to possible input arrangements. * n!
Another application of counting ... lower bounds

How does height relate to number of leaves?

**Theorem**: There are at most $2^h$ leaves in a binary tree with height $h$.

Which of the following is true?

A. It's possible to have a binary tree with height 3 and 1 leaf.
B. It's possible to have a binary tree with height 1 and 3 leaves.
C. Every binary tree with height 3 has 1 leaf.
D. Every binary tree with height 1 has 3 leaves.
E. None of the above.
How does height relate to number of leaves?

**Theorem:** There are at most $2^h$ leaves in a binary tree with height $h$.

**Proof:** By induction on the height $h \geq 0$.

**Base case** WTS that there are at most $2^0$ leaves in a binary tree with height $h=0$.

*What trees have height 0?*
How does height relate to number of leaves?

**Theorem:** There are at most $2^h$ leaves in a binary tree with height $h$.

**Proof:** By induction on the height $h \geq 0$.

*Base case* WTS that there are at most $2^0$ leaves in a binary tree with height $h=0$.

If a binary tree has height 0, its only node is the root. In this case the root is also a (and the only) leaf node. So, the number of leaves is $1 = 2^0$ in the only possible tree with $h=0$. ☺️
Another application of counting … lower bounds

How does height relate to number of leaves?

**Theorem**: There are at most $2^h$ leaves in a binary tree with height $h$.

**Proof**: By induction on the height $h \geq 0$.

*Induction Step* Let $h$ be some integer $\geq 0$ and assume (as the **IH**) that there are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$. 
Another application of counting ... lower bounds

**Induction Step** Let $h$ be some integer $\geq 0$ and assume (as the IH) that

There are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$. Consider a tree $U$ with height $h+1$. How can we relate it to trees of height $h$ so that we can use IH?
**Induction Step** Let \( h \) be some integer \( \geq 0 \) and assume (as the IH) that

\[
\text{There are at most } 2^h \text{ leaves in a binary tree with height } h.
\]

WTS that there are at most \( 2^{h+1} \) leaves in a binary tree with height \( h+1 \).

Consider a tree \( U \) with height \( h+1 \). How can we relate it to trees of height \( h \) so that we can use IH?

**Remove** all the leaves of \( U \). This gives a new tree, \( T \), of height \( h \).
By the IH the tree \( T \) has at most \( 2^h \) leaves.

To get from \( T \) to \( U \), we need to add back the leaves of \( U \). How many are there?
Another application of counting ... lower bounds

Induction Step Let $h$ be some integer $\geq 0$ and assume (as the IH) that

There are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$. Consider a tree $U$ with height $h+1$. How can we relate it to trees of height $h$ so that we can use IH?

Remove all the leaves of $U$. This gives a new tree, $T$, of height $h$. By the IH the tree $T$ has at most $2^h$ leaves.

To get from $T$ to $U$, we need to add back the leaves of $U$. How many are there? For each leaf of $T$, there are at most 2 leaves in $U$. 

Another application of counting ... lower bounds

**Induction Step** Let $h$ be some integer $\geq 0$ and assume (as the **IH**) that

There are at most $2^h$ leaves in a binary tree with height $h$.

WTS that there are at most $2^{h+1}$ leaves in a binary tree with height $h+1$.

Consider a tree $U$ with height $h+1$.  

**How can we relate it to trees of height $h$ so that we can use IH?**

**Remove** all the leaves of $U$.  This gives a new tree, $T$, of height $h$.

By the **IH** the tree $T$ has at most $2^h$ leaves.

To get from $T$ to $U$, we need to add back the leaves of $U$.  **How many are there?**

For each leaf of $T$, there are **at most 2 leaves** in $U$.

\[
\# \text{ leaves in } U \leq 2(\# \text{ leaves in } T) \leq 2(2^h) = 2^{h+1}
\]
Another application of counting … lower bounds

*What's the fastest possible worst case* for any sorting algorithm?

Maximum number of comparisons for algorithm is *height* of its tree diagram.

For any algorithm, what would be *smallest possible height*?

What do we know about the tree?

* Internal nodes correspond to comparisons.  
  * Leaves correspond to possible input arrangements.  

Each tree diagram must have at least *n! leaves*, so its height must be at least $\log_2(n!)$.  

* Depends on algorithm.  

$n!$
What's the **fastest possible worst case** for any sorting algorithm?

Maximum number of comparisons for algorithm is **height** of its tree diagram.

For any algorithm, what would be **smallest possible height**?

What do we know about the tree?
- Internal nodes correspond to comparisons. *Depends on algorithm.*
- Leaves correspond to possible input arrangements. $n!$

Each tree diagram must have at least $n!$ leaves, so its height must be at least $\log_2(n!)$. 

I.e. fastest possible worst case performance of sorting is $\log_2(n!)$.
Another application of counting … lower bounds

What's the fastest possible worst case for any sorting algorithm? \( \log_2(n!) \)

How big is that?

**Lemma:** For \( n > 1 \),

\[
\left(\frac{n}{2}\right)^{\frac{n}{2}} < n! < n^n
\]

**Proof:**

\[
n! = (n)(n-1)(n-2)\ldots\left(\frac{n}{2}\right)\ldots(3)(2)(1)
\]

\[
> \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\ldots\left(\frac{n}{2}\right)
\]

\[
= \left(\frac{n}{2}\right)^{\frac{n}{2}}
\]

\[
n! = (n)(n-1)(n-2)\ldots(3)(2)(1)
\]

\[
< (n)(n)(n)\ldots(n)(n)(n)
\]

\[
= n^n
\]
Another application of counting … lower bounds

What's the fastest possible worst case for any sorting algorithm? \( \log_2(n!) \)

How big is that?

Lemma: for \( n>1 \), \( \left( \frac{n}{2} \right)^{\frac{n}{2}} < n! < n^n \)

Theorem: \( \log_2(n!) \) is in \( \Theta(n \log n) \)

Proof: For \( n>1 \), taking logarithm of both sides in lemma gives

\[
\frac{n}{2} \log \left( \frac{n}{2} \right) < \log_2(n!) < n \log n
\]

i.e.

\[
\frac{1}{2} (n \log n - n \log 2) < \log_2(n!) < n \log n
\]
What's the **fastest possible worst case** for any sorting algorithm? \( \log_2(n!) \)

**How big is that?**

**Lemma:** for \( n > 1 \), \( \left( \frac{n}{2} \right)^{\frac{n}{2}} < n! < n^n \)

**Theorem:** \( \log_2(n!) \) is in \( \Theta(n \log n) \)

**Therefore,**
the best sorting algorithms will need \( \Theta(n \log n) \) comparisons in the worst case.

i.e. it's impossible to have a comparison-based algorithm that does better than **Merge Sort** (in the worst case).
Representing undirected graphs

**Strategy:**

1. **Count** the number of simple undirected graphs.
2. Compute **lower bound** on the number of bits required to represent these graphs.
3. Devise **algorithm** to represent graphs using this number of bits.

What's true about **simple undirected** graphs?

A. Self-loops are allowed.
B. Parallel edges are allowed.
C. There must be at least one vertex.
D. There must be at least one edge.
E. None of the above.

*Rosen p. 641-644*
In a simple undirected graph on n (labeled) vertices, how many edges are possible?

A. $n^2$
B. $n(n-1)$
C. $C(n,2)$
D. $2^{C(n,2)}$
E. None of the above.

** Recall notation: $C(n,k) = \binom{n}{k}$ **
In a simple undirected graph on n (labeled) vertices, how many edges are possible?

A. $n^2$
B. $n(n-1)$
C. $\binom{n}{2}$
D. $2^{\binom{n}{2}}$
E. None of the above.

Possibly one edge for each set of two distinct vertices.

** Recall notation: $\binom{n}{k} = \binom{n}{k}$ **
How many **different** simple undirected graphs on $n$ (labeled) vertices are there?

A. $n^2$
B. $n(n-1)$
C. $C(n,2)$
D. $2^{C(n,2)}$
E. None of the above.
How many different simple undirected graphs on n (labeled) vertices are there?

A. $n^2$
B. $n(n-1)$
C. $\binom{n}{2}$
D. $2^{\binom{n}{2}}$
E. None of the above.

For each possible edge, decide if in graph or not.

Conclude:
minimum number of bits to represent simple undirected graphs with n vertices is

$$\log_2(2^{\binom{n}{2}}) = \binom{n}{2} = \frac{n(n-1)}{2}$$
**Representing undirected graphs: Algorithm**

**Goal**: represent a simple undirected graph with n vertices using \( n(n-1)/2 \) bits.

**Idea**: store adjacency matrix, but since

- diagonal entries all zero (no self loops)
- matrix is symmetric (undirected graph)

only store the entries **above** the diagonal.

**How many entries of the adjacency matrix are above the diagonal?**

A. \( n^2 \)  
B. \( n(n-1) \)  
C. \( \text{C}(n,2) \)  
D. \( 2n \)  
E. None of the above.
Representing undirected graphs: Algorithm

**Goal:** represent a simple undirected graph with \( n \) vertices using \( n(n-1)/2 \) bits

**Idea:** store adjacency matrix, but since

- diagonal entries all zero \( \text{no self loops} \)
- matrix is symmetric \( \text{undirected graph} \)

only store the entries **above** the diagonal.

Can be stored as

0111101100
which uses \( C(5,2) = 10 \) bits.
What simple undirected graph is encoded by the binary string
011010 110101 111111 000000 110101 110010 ?

A. \[
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

B. \[
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

C. Either one of the above.
D. Neither one of the above.
In a simple directed graph on $n$ (labeled) vertices, how many edges are possible?

A. $n^2$
B. $n(n-1)$
C. $C(n,2)$
D. $2^{C(n,2)}$
E. None of the above.

Simple graph: no self loops, no parallel edges.
Representing *directed* graphs: Counting

In a simple *directed* graph on $n$ (labeled) vertices, how many edges are possible?

A. $n^2$
B. $n(n-1)$
C. $\binom{n}{2}$
D. $2\binom{n}{2}$
E. None of the above.

Choose starting vertex, choose ending vertex.

Simple graph: no self loops, no parallel edges.
Representing directed graphs: Counting

How many different simple directed graphs on n (labeled) vertices are there?

A. $n^2$
B. $n(n-1)$
C. $C(n,2)$
D. $2^{C(n,2)}$
E. None of the above.
Another way of counting that there are \(2^{n(n-1)}\) simple directed graphs with \(n\) vertices:

Represent a graph by

For each of the \(\binom{n}{2}\) pairs of distinct vertices \(\{v,w\}\), specify whether there is

* no edge between them
* an edge from \(v\) to \(w\) but no edge from \(w\) to \(v\)
* an edge from \(w\) to \(v\) but no edge from \(v\) to \(w\)
* edges both from \(v\) to \(w\) and from \(v\) to \(w\).
Another way of counting that there are $2^{n(n-1)}$ directed graphs with $n$ vertices:

Represent a graph by

For each of the $\binom{n}{2}$ pairs of distinct vertices $\{v,w\}$, specify whether there is

* no edge between them
* an edge from $v$ to $w$ but no edge from $w$ to $v$
* an edge from $w$ to $v$ but no edge from $v$ to $w$
* edges both from $v$ to $w$ and from $v$ to $w$.

Product rule!

$$4 \times 4 \times \ldots \times 4 = 4^{\binom{n}{2}} = 4^{n(n-1)/2} = 2^{n(n-1)}$$
Representing directed graphs: Lower bound

Conclude:
minimum number of bits to represent simple directed graphs with n vertices is
\[ \log_2(2^{n(n-1)}) = n(n-1) \]
Representing directed graphs: Algorithm

Encoding:
For each of the n vertices, indicate which of the other vertices it has an edge to.

How would you encode this graph using bits (0s and 1s)?

A. 123232443
B. 0110 0000 0101 0010
C. 110 000 011 001
D. None of the above.
Decoding:

Given a string of 0s and 1s of length $n(n-1)$,

- Define vertex set $\{1, \ldots, n\}$.
- First $n-1$ bits indicate edges from vertex 1 to other vertices.
- Next $n-1$ bits indicate edges from vertex 2 to other vertices.
- etc.

What graph does this binary string encode? 0110 1001 0001 1011 0100