Today's learning goals

• Define and differentiate between important sets
• Use correct notation when describing sets: {...}, intervals
• Define and prove properties of: subset relation, power set, Cartesian products of sets, union of sets, intersection of sets, disjoint sets, set differences, complement of a set
• Describe computer representation of sets with bitstrings
• Define and compute the cardinality of finite sets
Some definitions

Set: unordered collection of elements

\( \mathbb{N} \): natural numbers \( \{0, 1, 2, 3, \ldots\} \)
\( \mathbb{Z} \): integers \( \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
\( \mathbb{Z}^+ \): positive integers \( \{1, 2, 3, \ldots\} \)
\( \mathbb{Q} \): rational numbers \( \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\} \)
\( \mathbb{R} \): real numbers
\( \mathbb{R}^+ \): positive real numbers
\( \mathbb{C} \): complex numbers

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Arrows in set builder notation indicate "and"
Some definitions

Subset: \( A \subseteq B \) means \( \forall x ( x \in A \rightarrow x \in B ) \)

Rosen Sections 2.1, 2.2
Some definitions

**Subset:** $A \subseteq B$ means $\forall x (x \in A \rightarrow x \in B)$

**Theorem:** $\mathbb{Z}$ is a subset of $\mathbb{Q}$. 

*Rosen Sections 2.1, 2.2*
Some definitions

**Subset:** $A \subseteq B$ means $\forall x (x \in A \rightarrow x \in B)$

How would you prove that $\mathbb{R}$ is not a subset of $\mathbb{Q}$?
A. Prove that every real number is not rational.
B. Prove that every rational number is real.
C. Prove that there is a real number that is rational.
D. Prove that there is a real number that is not rational.
E. Prove that there is a rational number that is not real.
An (ir)rational excursion

**Theorem:** \( \mathbb{R} \) is not a subset of \( \mathbb{Q} \).

**Lemma:** \( \sqrt{2} \) is not rational.

**Corollary:** There are irrational numbers \( x, y \) such that \( x^y \) is rational.
An (ir)rational excursion

Theorem: \( \mathbb{R} \) is not a subset of \( \mathbb{Q} \).

Lemma: \( \sqrt{2} \) is not rational.

Recall: \( \mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\} \)

Details of proof in page 86: use contradiction!

Corollary: There are irrational numbers \( x, y \) such that \( x^y \) is rational.

Existential statement: can we build a witness?
An (ir)rational excursion

Theorem: $\mathbb{R}$ is not a subset of $\mathbb{Q}$.

Lemma: $\sqrt{2}$ is not rational.

Corollary: There are irrational numbers $x,y$ such that $x^y$ is rational.
Some definitions

Empty set: \( \emptyset = \{\} = \{x : x \neq x\} \)

Which of the following is not equal to the rest?

A. \(\{\}\)
B. \(\{\emptyset\}\)
C. \(\emptyset\)
D. \(\{x \in \mathbb{Z} \mid x > x^2\}\)
E. \(\{x \mid x \in \emptyset\}\)
Operations on sets

Power set: For a set S, its power set is the set of all subsets of S.

\[ \mathcal{P}(S) = \{ A \mid A \subseteq S \} \]

Which of the following is not true (in general)?

A. \( S \in \mathcal{P}(S) \)
B. \( \emptyset \in \mathcal{P}(S) \)
C. \( S \subseteq \mathcal{P}(S) \)
D. \( \emptyset \subseteq \mathcal{P}(S) \)
E. \( \emptyset \in S \)
Operations on sets

Given two sets A, B we can define

- Intersection of A and B: $A \cap B$
- Union of A and B: $A \cup B$
- Difference of A and B: $A - B$
- Cartesian product of A and B: $A \times B$
Operations on sets

Rosen Sections 2.1, 2.2

• Given two sets $A, B$ we can define

\[ A \cap B = \{ x \mid x \in A \land x \in B \} \]
\[ A \cup B = \{ x \mid x \in A \lor x \in B \} \]
\[ A - B = \{ x \mid x \in A \land x \notin B \} \]
\[ A \times B = \{ (x, y) \mid x \in A \land y \in B \} \]
Operations on sets

• Given two sets A, B we can define

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

$$A - B = \{x \mid x \in A \land x \notin B\}$$

$$A \times B = \{(x, y) \mid x \in A \land y \in B\}$$

Which of the following can't be labelled in this Venn diagram?
A. $A \cap B$
B. $A \cup B$
C. $A - B$
D. $A \times B$
E. None of the above.
Operations on sets

- Given two sets $A$, $B$ we can define

\[ A \cap B = \{x \mid x \in A \land x \in B\} \]
\[ A \cup B = \{x \mid x \in A \lor x \in B\} \]
\[ A - B = \{x \mid x \in A \land x \notin B\} \]
\[ A \times B = \{(x, y) \mid x \in A \land y \in B\} \]

Which of these is true?

A. $A \cap B = B \cap A$
B. $A \cup B = B \cup A$
C. $A - B = B - A$
D. $A \times B = B \times A$
E. None of the above.
Sizes of sets

- If $S$ is a set with exactly $n$ distinct elements, with $n$ a nonnegative integer, then $S$ is finite set and $|S| = n$.

Which of the following sets are finite?
Assume universe is set of real numbers.
A. $\emptyset$
B. $\mathbb{Q} \cap \{x \mid 0 \leq x \leq 1\}$
C. $\mathbb{Z} \cap \{x \mid 0 \leq x \leq 1\}$
D. $\mathbb{Z} \cup \{x \mid 0 \leq x \leq 1\}$
E. None of the above.
Operations on sets

- If the sets $A$, $B$ are finite then

$$|A \times B| = |A| \cdot |B|$$

$|B|$ many elements for each of the $|A|$ many elements in $A$
Operations on sets

- If the sets A, B are finite then

\[ |A \cup B| = ? \]

A. \(|A| + |B|\)
B. \(|A| - |B|\)
C. \(|A| |B|\)
D. \(|A|^{|B|}\)
E. None of the above.
Operations on sets

- If the sets $A$, $B$ are finite then

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

What's the size of the difference $A-B$?

A. $|A \cup B| - |A \cap B|
B. $|A \cup B| - |A| - |B|
C. $|A| - |A \cap B|
D. $|A| - |B|
E. None of the above.
Operations on sets

• Two sets A and B are disjoint iff

\[ A \cap B = \emptyset \]

Which of the following is not an equivalent characterization of A and B being disjoint?

A. \(|A \cup B| = |A| + |B|\)
B. \(A - B = A\)
C. \(A \subseteq B\)
D. \(\neg \exists x (x \in A \land x \in B)\)
E. None of the above.
Operations on sets

- If the set $A$ is finite then

  $$|\mathcal{P}(A)| = ?$$

Does the power set of $A$ depend just on the size of $A$?

Does the size of the power set of $A$ depend just on the size of $A$?
Operations on sets

If the set $A$ is finite then

$$|\mathcal{P}(A)| = ?$$

Does the power set of $A$ depend just on the size of $A$?

Does the size of the power set of $A$ depend just on the size of $A$?

$$\mathcal{P}\left(\{2\}\right) = \{\emptyset, \{2\}\}$$
$$\mathcal{P}\left(\{8\}\right) = \{\emptyset, \{8\}\}$$
$$\mathcal{P}\left(\{2, 5\}\right) = \{\emptyset, \{2\}, \{5\}, \{2, 5\}\}$$
$$\mathcal{P}\left(\{2, 8\}\right) = \{\emptyset, \{2\}, \{8\}, \{2, 8\}\}$$
$$\mathcal{P}\left(\{2, 5, 8\}\right) = \{\emptyset, \{2\}, \{5\}, \{8\}, \{2, 5\}, \{2, 8\}, \{5, 8\}, \{2, 5, 8\}\}$$
Set of home network components:
{ server, switch, workstation, wifi, iPhone, laptop, Smartphone, Desktop Roommate1, Desktop Roommate2, Desktop Roommate3}
Representing sets

Set of home network components:
{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
Representing sets

Rosen p. 134

Set of home network components: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

How to represent the subset which contains all the desktop PCs?

A. {8, 9, 10}
B. {10, 8, 9}
C. {8, 8, 9, 10}
D. None of the above.
E. All of the above.
Representing sets

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Set of home network components:
{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Alternatively: using bit strings

first bit is 0 if item 1 (server) is not in the set; 1 if is
second bit is 0 if item 2 (switch) is not in the set; 1 if is

How to represent the subset which contains all the desktop PCs using bit strings?
A. 0000000000
B. 0000000111
C. 1110000000
D. None of the above.
E. All of the above.
Assume that universal set \( U \) is finite of size \( n \) (and \( n \) is not too big). Specify arbitrary ordering of elements of \( U \): \( a_1, a_2, a_3, \ldots, a_n \).

Represent subset \( A \) of \( U \) by the bit string where, for each \( i \), the \( i \)th bit in the bit string is 1 if \( a_i \) is an element of \( A \) and it's 0 if \( a_i \) is not an element of \( A \).

Describe, using set operations, the set described by the bit string which results from taking bit string for \( A \) and flipping each bit 0 \( \rightarrow \) 1, 1 \( \rightarrow \) 0.

A. The power set of \( A \), \( \mathcal{P}(A) \)  
B. The union of \( A \) with itself, \( A \cup A \)  
C. The difference \( U - A \)  
D. The Cartesian product \( A \times A \)  
E. None of the above.
Operations on sets

- If the set $A$ is finite then

$$|\mathcal{P}(A)| = ?$$

*How many subsets of $A$ are there?*

Represent subset $A$ of $U$ by the bit string where, for each $i$, the $i$th bit in the bit string is 1 if $a_i$ is an element of $A$ and it's 0 if $a_i$ is not an element of $A$.

*How many different substrings are there?*
Operations on sets

• If the set $A$ is finite then

$$|\mathcal{P}(A)| = 2^{|A|}$$

How many subsets of $A$ are there?

Represent subset $A$ of $U$ by the bit string where, for each $i$, the $i$th bit in the bit string is 1 if $a_i$ is an element of $A$ and it's 0 if $a_i$ is not an element of $A.$

How many different substrings are there?

Rosen Sections 2.1, 2.2
Next up

- How do we prove these general formulas?
- Induction!