Background material

Relations

- A relation over a set S is a set \( R \subseteq S \times S \)
  - We write \( a R b \) for \((a, b) \in R\)
- A relation \( R \) is:
  - reflexive iff
    \[ \forall a \in S : a R a \]
  - transitive iff
    \[ \forall a \in S, b \in S, c \in S : a R b \land b R c \Rightarrow a R c \]
  - symmetric iff
    \[ \forall a, b \in S : a R b \Rightarrow b R a \]
  - anti-symmetric iff
    \[ \forall a, b \in S : a R b \Rightarrow \neg (b R a) \]

Partial orders

- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:

Partial orders

- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:
- A partially ordered set (a poset) is a pair \((S, \leq)\) of
  a set \(S\) and a partial order \(\leq\) over the set
- Examples of posets: \((2^S, \subseteq), (Z, \leq), (Z, \text{divides})\)

Lub and glb

- Given a poset \( (S, \leq)\), and two elements \( a \in S \) and \( b \in S \), then the:
  - least upper bound (lub) is an element \( c \) such that
    \[ a \leq c, b \leq c, \text{ and } \forall d \in S : (a \leq d \land b \leq d) \Rightarrow c \leq d \]
  - greatest lower bound (glb) is an element \( c \) such that
    \[ c \leq a, c \leq b, \text{ and } \forall d \in S : (d \leq a \land d \leq b) \Rightarrow d \leq c \]
Lub and glb

• Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  – least upper bound (lub) is an element $c$ such that $a \leq c$, $b \leq c$, and $\forall d \in S : (a \leq d \land b \leq d) \Rightarrow c \leq d$
  – greatest lower bound (glb) is an element $c$ such that $c \leq a$, $c \leq b$, and $\forall d \in S : (d \leq a \land d \leq b) \Rightarrow d \leq c$
• lub and glb don’t always exists:

Lattices

• A lattice is a tuple $(S, \sqsubseteq, \sqcup, \sqcap, 0, 1)$ such that:
  – $(S, \sqsubseteq)$ is a poset
  – $\forall a \in S : \bot \sqsubseteq a$
  – $\forall a \in S : a \sqsubseteq \top$
  – Every two elements from $S$ have a lub and a glb
  – $\sqcup$ is the least upper bound operator, called a join
  – $\sqcap$ is the greatest lower bound operator, called a meet

Examples of lattices

• Powerset lattice

Examples of lattices

• Boolean expressions
Examples of lattices

- Booleans expressions

Examples of lattices

- Booleans expressions

Examples of lattices

- Booleans expressions