1. (15 + 5 marks)

(a) Prove that the following expressions are equivalent:

   (a) \( p \rightarrow (q \lor r) \equiv (p \land \neg q) \rightarrow r \)
   
   (b) \( (p \land \neg r) \rightarrow q \)

(ba) (morning class) If the above propositions are TRUE and \( p \) is TRUE and \( q \) is TRUE then what can you deduce about \( r \).

(bb) (noon class) If the above propositions are TRUE and \( p \) is TRUE and \( q \) is FALSE then what can you deduce about \( r \).

(a). Solution #1 (truth table): We have 3 variables: \( p, q, r \). Each of them can take 2 values: TRUE and FALSE. So in total, we have \( 2^3 = 8 \) cases to check (Use T and F for TRUE and FALSE, respectively):

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( (q \lor r) )</th>
<th>( (p \land \neg q) )</th>
<th>( (p \land \neg r) \rightarrow r )</th>
<th>( (p \land \neg r) \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
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<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

As shown from the truth table, \( p \rightarrow (q \lor r), (p \land \neg q) \rightarrow r, (p \land \neg r) \rightarrow q \) take the same value for every possible assignment of \( p, q, r \). Therefore, they are equivalent.

Solution #2 (equivalent derivations by logical laws): The intuition is to eliminate \( \rightarrow \) and represent an expression in \( \land, \lor, \neg \) only. We have
(i) $p \rightarrow (q \lor r) \iff \neg p \lor (q \lor r)$  
\hspace{10pt} (Page 31 of the text)  
\hspace{10pt} $\iff (\neg p \lor q) \lor r$  
\hspace{10pt} (Associative Law)

(ii) $(p \land \neg q) \rightarrow r \iff \neg (p \land \neg q) \lor r$  
\hspace{10pt} (Page 31 of the text)  
\hspace{10pt} $\iff (\neg p \lor \neg \neg q) \lor r$  
\hspace{10pt} (De Morgan’s Law)  
\hspace{10pt} $\iff (\neg p \lor q) \lor r$  
\hspace{10pt} (Double negation Law)

(iii) $(p \land \neg r) \rightarrow q \iff \neg (p \land \neg r) \lor q$  
\hspace{10pt} (Page 31 of the text)  
\hspace{10pt} $\iff (\neg p \lor \neg r) \lor q$  
\hspace{10pt} (De Morgan’s Law)  
\hspace{10pt} $\iff (\neg p \lor r) \lor q$  
\hspace{10pt} (Double negation Law)  
\hspace{10pt} $\iff \neg p \lor (r \lor q)$  
\hspace{10pt} (Associative Law)  
\hspace{10pt} $\iff \neg p \lor (q \lor r)$  
\hspace{10pt} (Commutative Law)  
\hspace{10pt} $\iff (\neg p \lor q) \lor r$  
\hspace{10pt} (Associative Law)

Hence, we see that $p \rightarrow (q \lor r)$, $(p \land \neg q) \rightarrow r$, $(p \land \neg r) \rightarrow q$ can be transformed into the same form: $(\neg p \lor q) \lor r$ through equivalent derivations. Therefore, they are equivalent. ■

(ba). Look at the truth table, we see that when $p = \text{TRUE}$, $q = \text{TRUE}$, and all three implications are TRUE, $r$ can be either TRUE or FALSE. ■

(bb). Look at the truth table, we see that when $p = \text{TRUE}$, $q = \text{FALSE}$, and all three implications are TRUE, $r$ must be TRUE. ■
2. (5 + 5 + 5 marks) Express these system specifications using the propositions:

\[ p = \text{“The message is canned for viruses”} \]
\[ q = \text{“The message was sent from an unknown system”} \]

together with logical connectives.

(a) The message is scanned for viruses whenever the message was sent from an unknown system.

(b) The message was sent from an unknown system but it was not scanned for viruses.

(c) When a message is not sent from an unknown system, it is not scanned for viruses.

(a) “whenever”. Put equivalently, the statement is:

When the message was sent from an unknown system, the message is scanned for viruses.

\[ q \rightarrow p. \]

(b) “and”. \( q \land \neg p. \)

(c) “when”. \( \neg q \rightarrow \neg p \)

3. (15 marks) Prove or disprove that in the above problem (Statement (a)) = \( \neg (\text{Statement (b)}) \). Give reasons.

We want to prove \( q \rightarrow p = \neg(q \land \neg p) \). Let’s look at the truth table of LHS and RHS:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( q \rightarrow p )</th>
<th>( q \land \neg p )</th>
<th>( \neg(q \land \neg p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
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<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

As shown from the truth table, \( p \rightarrow q, \neg(q \land \neg p) \) take the same value for every possible assignment of \( p, q \). Therefore, they are equivalent.