1. Show that \( a^2 + a^4 \equiv 0 \pmod{5} \) if \( a \equiv 2 \pmod{5} \) or \( a \equiv 3 \pmod{5} \) or if 5 divides \( a \).

**Proof.** We want to show that if \( a \equiv 2 \pmod{5} \) or \( a \equiv 3 \pmod{5} \) or if 5 divides \( a \), then \( a^2 + a^4 \equiv 0 \pmod{5} \). In other words, we want to show that \( a^2 + a^4 = 5k + 0 \), where \( k \) is an integer. We will do this by considering each of the three cases separately.

First let’s note some theorems about modular arithmetic that will help us with our proof (Theorem 4, NT-7):

- If \( x \equiv m \pmod{d} \) and \( y \equiv n \pmod{d} \), then \( x + y \equiv m + n \pmod{d} \).
- If \( x \equiv m \pmod{d} \), then \( x^n \equiv m^n \pmod{d} \).

**Case 1:** \( a \equiv 2 \pmod{5} \)

Since \( a \equiv 2 \pmod{5} \), then \( a^2 \equiv 2^2 \pmod{5} \) and \( a^4 \equiv 2^4 \pmod{5} \). \( a^2 + a^4 \equiv 2^2 + 2^4 \pmod{5} \equiv 20 \pmod{5} \). By the definition of mod, we can write \( a^2 + a^4 = 5b + 20 \) where \( b \) is an integer, or \( a^2 + a^4 = 5(b + 4) + 0 \). Since \( b \) is an integer, \( b + 4 \) must also be an integer. Therefore, if \( a \equiv 2 \pmod{5} \), then \( a^2 + a^4 \equiv 0 \pmod{5} \).

**Case 2:** \( a \equiv 3 \pmod{5} \)

Since \( a \equiv 3 \pmod{5} \), then \( a^2 \equiv 3^2 \pmod{5} \) and \( a^4 \equiv 3^4 \pmod{5} \). \( a^2 + a^4 \equiv 3^2 + 3^4 \pmod{5} \equiv 90 \pmod{5} \). By the definition of mod, we can write \( a^2 + a^4 = 5c + 90 \) where \( c \) is an integer, or \( a^2 + a^4 = 5(c + 18) + 0 \). Since \( c \) is an integer, \( c + 18 \) must also be an integer. Therefore, if \( a \equiv 3 \pmod{5} \), then \( a^2 + a^4 \equiv 0 \pmod{5} \).

**Case 3:** 5 divides \( a \)

By the definition of divides, \( a = 5d \), where \( d \) is an integer. So, \( a^2 + a^4 = (5d)^2 + (5d)^4 = 25d^2 + 625d^4 = 5(5d^2 + 125d^4) + 0 \). Since \( d \) is an integer, \( 5d^2 + 125d^4 \) must also be an integer. Therefore, if 5 divides \( a \), then \( a^2 + a^4 \equiv 0 \pmod{5} \).

2. Let \( x \) be an integer. Prove that if \( x^2 - 6x + 5 \) is even then \( x \) must be odd.

**Proof.** Assume for the sake of contradiction that \( x^2 - 6x + 5 \) is even and \( x \) is even. By definition of even, \( x = 2c \) where \( c \) is an integer. By substitution, \( x^2 - 6x + 5 = (2c)^2 - 6(2c) + 5 = 4c^2 - 12c + 5 = 4c^2 - 12c + 4 + 1 = 2(2c^2 - 6c + 2) + 1 \). Since \( 2c^2 - 6c + 2 \) is an integer, \( x^2 - 6x + 5 \) must be odd, which is a contradiction.
3. Show that for any posetive number \(a\) and \(b\),

\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

**Proof.**

\[
\frac{a + b}{2} \geq \sqrt{ab} \iff \left(\frac{a + b}{2}\right)^2 \geq ab
\]

\[
\iff (a + b)^2 = 4ab \iff a^2 + b^2 + 2ab \geq 4ab
\]

\[
\iff a^2 - 2ab + b^2 \geq 0 \iff (a - b)^2 \geq 0
\]

Since square on any number is greater than zero so the \((a - b)^2 \geq 0\) and so we have the inequality.

4. If \(a\) is an odd integer prove that \(a^2 - 1\) is always divisible by 8.

**Proof.** If \(a\) is divided by 4 then the possible set of remainders are 0, 1, 2 and 3.

Since \(a\) is odd so the remainder when divided by 4 cannot be 0 or 2. So if \(a\) is an odd integer then the remainder when divided by 4 is 1 or 3.

Now we do case analysis:

**Case 1:** Let the remainder be 1. So \(a = 4k + 1\) for some integer \(k\). Thus

\[
a^2 = (4k + 1)^2 = 16k^2 + 8k + 1 = 8(2k^2 + k) + 1
\]

So \(a^2 - 1 = 8(2k^2 + k)\) and so \(a^2 - 1\) is divisible by 8.

**Case 2:** Let the remainder be 3. So \(a = 4k + 3\) for some integer \(k\). Thus

\[
a^2 = (4k + 3)^2 = 16k^2 + 24k + 9 = 8(2k^2 + 3k + 1) + 1
\]

So \(a^2 - 1 = 8(2k^2 + 3k + 1)\) and so \(a^2 - 1\) is divisible by 8.

5. Prove that if \(k\) and \(\ell\) are posetive integers then \(k^2 - \ell^2\) can never be equal to 2.

**Proof.** If \(a^2 - b^2\) has to be 2 then either both \(a\) and \(b\) has to be even or both have to be odd. If one is even and the other is odd then \(a^2 - b^2\) would be odd.

Now we solve case wise depending on whether both are odd or both are even.
Case 1: Both $a$ and $b$ are even.
Then say $a = 2m$ and $b = 2n$ when $m$ and $n$ are positive integers.
Then $a^2 - b^2 = (2m)^2 - (2n)^2 = 4m^2 - 4n^2 = 4(m^2 - n^2)$.
Now since $(m^2 - n^2)$ is an integer and 4 times an integer cannot be 2 so $a^2 - b^2$ cannot be 2 in this case.

Case 2: Both $a$ and $b$ are odd.
Then say $a = 2m + 1$ and $b = 2n + 1$ when $m$ and $n$ are positive integers. Then
\[ a^2 - b^2 = (2m + 1)^2 - (2n + 1)^2 = 4m^2 + 4m + 1 - 4n^2 - 4n - 1 = 4(m^2 + m - n^2 - n). \]
Now since $(m^2 + m - n^2 - n)$ is an integer and 4 times an integer cannot be 2 so $a^2 - b^2$ cannot be 2 in this case.

6. Prove that there are infinitely many primes of the form $6k + 5$. That is, consider the primes which has a remainder 5 when divided by 6. Prove that there are infinitely many such primes.

Solution.

— Left as challenge question —