Problem 1
Consider the problem of determining if the language of a Turing machine $M$ is closed under reverse, i.e., given a Turing machine $M$, determine if the language $L = \mathcal{L}(M)$ satisfies $w^r \in L$ for all $w \in L$. (Here $w^r$ is the string reversal operations, that given a string $w = w_1w_2 \ldots w_n$ of length $n$, produces the string $w^r = w_nw_{n-1} \ldots w_1$.

Formulate the above problem as a language, and prove that it is neither recognizable nor co-recognizable.

Problem 2
Consider the following program analysis problem: given a program (written in Java, or any other imperative programming language of your choice) containing a global variable $x$, determine if the value of $x$ can ever exceed a given bound $B$. Such problem is of interest in many settings, where for example

- $x$ is used as an index into an array, and you want to determine that $x$ never travels beyond the array boundaries.
- $x$ is a numerical quantity, and you want to determine that $x$ can be stored within a given number of bits (e.g., using a 32 or 64 bits register.)

Prove that the above problem in undecidable, i.e., there exists no computer program (or Turing machine) $P$ that on input another computer program (or Turing machine) $Q$, a variable name $x$ and a bound $B$, always terminates and outputs “yes” if and only if there exists an input $w$ such that when $Q$ is run on input $w$, at some point during the computation the variable $x$ will be assigned a value larger than $B$.

Problem 3
Consider the following languages

$L_1 = \{\langle M \rangle : M \text{ is a decider} \}$
$L_2 = \{\langle M \rangle : M \text{ is a TM such that } \mathcal{L}(M) \text{ is decidable} \}$
$L_3 = \{\langle M \rangle : M \text{ is a TM such that } \mathcal{L}(M) \text{ is recognizable} \}$
$L_4 = \{\langle M \rangle : M \text{ is a TM such that } \mathcal{L}(M) \text{ is corecognizable} \}$

Answer the following questions:

1. Show that at least one language $L_4$ is decidable.
2. Show that at least one language $L_4$ is undecidable.
3. Two languages from $L_1, L_2, L_3, L_4$ are equal. Which two? Justify your answer.