CSE 20: Lecture 7
Boolean Algebra

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Outline

1. Introduction
2. Definitions
   - Interpretation of Set Operations
   - Interpretation of Logic Operations
3. Theorems and Proofs
   - Multi-valued Boolean Algebra
4. Transformations
1. Introduction: iClicker

Boolean algebra can be used for:
A. Set operation
B. Logic operation
C. Software verification
D. Hardware designs
E. All of the above.
1. Introduction

Boolean algebra can be used for:
A. Set operation (union, intersect, exclusion)
B. Logic operation (AND, OR, NOT)
C. Software verification
D. Hardware designs (control, data process)
Introduction: Basic Components

We use binary bits to represent true or false.
A=1: A is true
A=0: A is false
We use AND, OR, NOT gates to operate the logic.

NOT gate inverts the value (flip 0 and 1)
y = NOT (A) = A’

<table>
<thead>
<tr>
<th>id</th>
<th>A</th>
<th>NOT A</th>
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<tbody>
<tr>
<td>0</td>
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<td>1</td>
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<tr>
<td>1</td>
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<td>0</td>
</tr>
</tbody>
</table>
Introduction: Basic Components

OR gate: Output is true if either input is true

\[ y = A \text{ OR } B \]

<table>
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<tr>
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<th>B</th>
<th>A OR B</th>
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<tbody>
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<td>0</td>
<td>1</td>
<td>1</td>
</tr>
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<td>1</td>
<td>0</td>
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<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</table>
Introduction: Basic Components

AND gate: Output is true only if all inputs are true

\[ y = A \text{ AND } B \]

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<th>B</th>
<th>A \text{ AND } B</th>
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Introduction: Half Adder

A Half Adder:
Carry = A AND B
Sum = (A AND B’’) OR (A’ AND B)

<table>
<thead>
<tr>
<th>id</th>
<th>A, B</th>
<th>C_{out}, S_{um}</th>
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<td>0 0</td>
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<tr>
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<td>01</td>
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<tr>
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<tr>
<td>3</td>
<td>11</td>
<td>1 0</td>
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</table>
Introduction: Half Adder

A Half Adder:

\[ C_{out} = A \text{ AND } B \]
\[ S_{um} = (A \text{ AND } B') \text{ OR } (A' \text{ AND } B) \]

\[ S_{um}: \]

\[ A \]
\[ A' \]
\[ A' \text{ and } B \]
\[ B' \]
\[ B \]
\[ (A \text{ and } B') \text{ or } (A' \text{ and } B) \]
Introduction: Multiplexer

A multiplexer:
If $S$ then $Z=A$ else $Z=B$

\[
(S \text{ and } A) \text{ or } (S' \text{ and } B)
\]
2. Definition

Boolean Algebra: A set of elements $B$ with two operations: $+$ (OR, $U$, $\lor$) and $*$ (AND, $\cap$, $\land$), satisfying the following 4 laws for every $a$, $b$, $c$ in $B$.

P1. Commutative Laws:
$$a + b = b + a; \quad a * b = b * a,$$

P2. Distributive Laws:
$$a + (b * c) = (a + b) * (a + c); \quad a * (b + c) = (a * b) + (a * c),$$

P3. Identity Elements: Set $B$ has two distinct elements denoted as 0 and 1, such that $a + 0 = a; \quad a * 1 = a$,

P4. Complement Laws:
$$a + a' = 1; \quad a * a' = 0.$$
Interpretation of Set Operations

- Set: Collection of Objects

- Example:
  - \( A = \{1, 3, 5, 7, 9\} \)
  - \( N = \{x | x \text{ is a positive integer}\}, \text{ e.g. } \{1, 2, 3, \ldots\} \)
  - \( Z = \{x | x \text{ is an integer}\}, \text{ e.g. } \{-1, 0, 4\} \)
  - \( Q = \{x | x \text{ is a rational number}\}, \text{ e.g. } \{-0.75, \frac{2}{3}, 100\} \)
  - \( R = \{x | x \text{ is a real number}\}, \text{ e.g. } \{\pi, 12, -\frac{1}{3}\} \)
  - \( C = \{x | x \text{ is a complex number}\}, \text{ e.g. } \{2 + 7i\} \)
  - \( \Phi = \{\} \) or empty set
P1. Commutative Laws in Venn Diagram

\[ A \cup B = B \cup A \quad \quad \quad A \cap B = B \cap A \]
P2. Distributive Laws

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
P2. Distributive Laws

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
P3. Identity Elements

- $0 = \emptyset$
- $1 = \text{Universe of the set}$
- $A \cup 0 = A$
- $A \cap 1 = A$
P4: Complement

- $A \cup A' = 1$
- $A \cap A' = 0$