CSE20 Lecture 3
Number Systems

2. Binary Numbers
3. Gray Code
4. Negative Numbers

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Outlines

1. Goal of the Negative Number Systems
2. Definition
   2. 1’s Complement Rep.
   3. 2’s Complement Rep.
3. Arithmetic Operations
4.1 Goal of Negative Number System: iClicker

The goal of negative number system is

• A. to maximize the range of the numbers
• B. to simplify the hardware implementation
• C. to improve human interface
• D. All of the above.
4.1 Goal of negative number systems

• Signed system: Simple. Just flip the sign bit
  • 0 = positive
  • 1 = negative

• One’s complement: Replace subtraction with addition
  — Easy to derive (Just flip every bit)

• Two’s complement: Replace subtraction with addition
  — Addition of one’s complement and one produces the two’s complement.
4.2 Definitions: Given a positive integer $x$, we represent $-x$

- **1’s complement:**
  - Formula: $2^n - 1 - x$
    - i.e. $n=4$, $2^4 - 1 - x = 15 - x$
    - In binary: $(1111) - (b_3 b_2 b_1 b_0)$
    - Just flip all the bits.

- **2’s complement:**
  - Formula: $2^n - x$
    - i.e. $n=4$, $2^4 - x = 16 - x$
    - In binary: $(10000) - (0 b_3 b_2 b_1 b_0)$
    - Just flip all the bits and add 1.
### 4.2 Definitions: 4-bit example, id vs. value

**Signed:** $b_3=1$, 1’s: $15-x$, 2’s: $16-x$

<table>
<thead>
<tr>
<th>id</th>
<th>$b_3b_2b_1b_0$</th>
<th>Signed</th>
<th>1’s</th>
<th>2’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>.</td>
<td>....</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>7</td>
<td>0111</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>-0</td>
<td>-7</td>
<td>-8</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td>-1</td>
<td>-6</td>
<td>-7</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>-2</td>
<td>-5</td>
<td>-6</td>
</tr>
<tr>
<td>11</td>
<td>1011</td>
<td>-3</td>
<td>-4</td>
<td>-5</td>
</tr>
<tr>
<td>12</td>
<td>1100</td>
<td>-4</td>
<td>-3</td>
<td>-4</td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
<td>-5</td>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>14</td>
<td>1110</td>
<td>-6</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>-7</td>
<td>-0</td>
<td>-1</td>
</tr>
</tbody>
</table>

15-x=id or id+x=15.
16-x=id or id+x=16.

$b_{n-1}=1$ for negative numbers
4.2 Definitions: 4-bit example, value vs. $b_3b_2b_1b_0$

<table>
<thead>
<tr>
<th>-x</th>
<th>signed</th>
<th>1’s</th>
<th>2’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0</td>
<td>1000</td>
<td>1111</td>
<td>0000</td>
</tr>
<tr>
<td>-1</td>
<td>1001</td>
<td>1110</td>
<td>1111</td>
</tr>
<tr>
<td>-2</td>
<td>1010</td>
<td>1101</td>
<td>1110</td>
</tr>
<tr>
<td>-3</td>
<td>1011</td>
<td>1100</td>
<td>1101</td>
</tr>
<tr>
<td>-4</td>
<td>1100</td>
<td>1011</td>
<td>1100</td>
</tr>
<tr>
<td>-5</td>
<td>1101</td>
<td>1010</td>
<td>1011</td>
</tr>
<tr>
<td>-6</td>
<td>1110</td>
<td>1001</td>
<td>1010</td>
</tr>
<tr>
<td>-7</td>
<td>1111</td>
<td>1000</td>
<td>1001</td>
</tr>
<tr>
<td>-8</td>
<td>NA</td>
<td>NA</td>
<td>1000</td>
</tr>
</tbody>
</table>
4.2 Definition: Example

Given bit width n=5 for x = 6 (00110)_2, we represent –x.

- Signed number: \((b_4b_3b_2b_1b_0)_2 = (10110)_2\)
- 1’s complement: \(2^5 - 1 - x = 32 - 1 - 6 = 25\)
  \((b_4b_3b_2b_1b_0)_2 = (11001)_2\)
- 2’s complement: \(2^5 - x = 32 - 6 = 26\)
  \((b_4b_3b_2b_1b_0)_2 = (11010)_2\)
4.2 Definition: iClicker

Given bit width n=5 for x= 11 (01011)₂, we represent –x in 1’s complement as

• A. (10100)₂
• B. (10101)₂
• C. (11010)₂
• D. None of the above.
4.2 Definitions: Examples
Given n-bits, what is the range of my numbers in each system?

• 3 bits:
  – Signed: -3, 3
  – 1’s: -3, 3
  – 2’s: -4, 3

• 6 bits
  – Signed: -31, 31
  – 1’s: -31, 31
  – 2’s: -32, 31

• 5 bits:
  – Signed: -15, 15
  – 1’s: -15, 15
  – 2’s: -16, 15

• Given 8 bits
  – Signed: -127, 127
  – 1’s: -127, 127
  – 2’s: -128, 127

**Formula for calculating the range**

Signed & 1’s: \((-2^{n-1} - 1), (2^{n-1} - 1)\)

2’s: \(-2^{n-1}, (2^{n-1} - 1)\)
4.3 Arithmetic Operation

- Conversion
- Addition and subtraction
- Inverse conversion
- Overflow
4.3 Arithmetic Operations: Conversion
Derivation of 1’s Complement

**Theorem 1:** For 1’s complement, given a positive number \((x_{n-1}, x_{n-2}, \ldots, x_0)_2\), the negative number is \((\bar{x}_{n-1}, \bar{x}_{n-2}, \ldots, \bar{x}_0)_2\) where \(\bar{x} = 1 - x\)

**Proof:**
(i). \(2^n - 1\) in binary is an n bit vector \((1,1, \ldots, 1)_2\)
(ii). \(2^n - 1 - x\) in binary is \((1,1, \ldots, 1)_2 - (x_{n-1}, x_{n-2}, \ldots, x_0)_2\).

The result is \((\bar{x}_{n-1}, \bar{x}_{n-2}, \ldots, \bar{x}_0)_2\)
4.3 Arithmetic Operations: Conversion
Derivation of 2’s Complement

**Theorem 2:** For 2’s complement, given a positive integer \( x \), the negative number is the sum of its 1’s complement and 1.

**Proof:** \( 2^n - x = 2^n - 1 - x + 1 \). From theorem 1, we have

\[
\left( \bar{x}_{n-1}, \bar{x}_{n-2}, \ldots, \bar{x}_0 \right)_2 + (0, 0, \ldots, 1)_2
\]
4.3 Arithmetic Operations: Conversion

Ex: n=5, x = 9 \((01001)_2\)
1’s complement: \(2^5-1-x= 32-1-9=22\)
   \(=(10110)_2\)
2’s complement: \(2^5-x=32-9=23\)
   \(=(10111)_2\)

Ex: n=5, x = 13 \((01101)_2\)
1’s complement: \(2^5-1-x= 32-1-13=18\)
   \(=(10010)_2\)
2’s complement: \(2^5-x=32-13=18\)
   \(=(10011)_2\)
4.3 Conversion: One’s Complement

Hardware:

\[ x_{n-1} \quad x_{n-2} \quad \ldots \quad x_0 \]

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Inverters
4.3 Addition and Subtraction:

Given two positive integers $x$ & $y$, we replace subtraction with complement conversion. Suppose the sum is valid in the form of the complement. Then we don’t need subtraction in hardware implementation.
### 4.3 Addition and Subtraction: 2’s Comp.

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>Addition in 2’s comp.</th>
<th>Solution in 2’s comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + y$</td>
<td>$x + y$</td>
<td>$x + y$</td>
</tr>
<tr>
<td>$x - y$</td>
<td>$x + (2^n - y)$</td>
<td>$2^n + (x - y)$ (x(&lt;y)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x - y$ (x(\geq y$)</td>
</tr>
<tr>
<td>$-x + y$</td>
<td>$(2^n - x) + y$</td>
<td>$2^n + (-x + y)$ (x(&gt; y$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-x + y$ (x(\leq y$)</td>
</tr>
<tr>
<td>$-x - y$</td>
<td>$(2^n - x) + (2^n - y)$</td>
<td>$2^n - x - y$</td>
</tr>
</tbody>
</table>

Note the similarity of the last two columns.
4.3 Addition and Subtraction: 2’s Comp.

Input: two positive integers x & y,
1. We represent the operands in two’s complement.
2. We sum up the two operands and ignore bit n.
3. The result is the solution in two’s complement.

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>Addition in 2’s comp.</th>
<th>Solution in 2’s comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>x - y</td>
<td>x + (2^n-y)=2^n+x-y</td>
<td>2^n+(x-y) (x&lt;y)</td>
</tr>
<tr>
<td></td>
<td>If x&lt;y, b_n=0</td>
<td>x-y (x&gt;=y)</td>
</tr>
<tr>
<td></td>
<td>Else, b_n=1</td>
<td></td>
</tr>
<tr>
<td>-x + y</td>
<td>(2^n-x) + y =2^n-x+y</td>
<td>2^n+(-x+y) (x&gt;y)</td>
</tr>
<tr>
<td></td>
<td>If x&gt;y, b_n=0</td>
<td>-x+y (x&lt;=y)</td>
</tr>
<tr>
<td></td>
<td>Else, b_n=1</td>
<td></td>
</tr>
<tr>
<td>-x - y</td>
<td>2^n+ (2^n-x-y)</td>
<td>2^n-x-y</td>
</tr>
<tr>
<td></td>
<td>b_n=1</td>
<td></td>
</tr>
</tbody>
</table>
4.3 Arithmetic Operations: 2’s comp.
Example: 4 – 3 = 1

In 2’s complement, we represent -3 as (1101)₂

\[
\begin{align*}
0100 \ (4) \\
+1101 \ (13=16-3) \\
\hline
10001 \ (17=16+1)
\end{align*}
\]

Formula: \( x + (2^n - y) = 4 + (16-3) = 16 + 1 \)
We discard the extra 1 at the left which is from 2’s complement of -3. Note that bit \( b_{n-1} \) is 0. Thus, the result is positive.
4.3 Arithmetic Operations: 2’s complement

Example: -4 +3 = -1

In 2’s comp., we represent -4 as \((1100)_2\)

\[
\begin{array}{c}
1100 \ (12=16-4) \\
+ 0011 \ (3) \\
\hline
1111 \ \Rightarrow \ -1 \text{ in 2’s comp.}
\end{array}
\]

Formula: \((2^n-x)+y=16-4+3=16-1=15 \ (-1 \text{ in 2’s comp.})\)

Note that \(b_{n-1}=1\). Thus, the solution is negative.
4.3 Arithmetic Operations: 2’s complement

Example: -4 -3 = -7

\[
\begin{align*}
1100 \ (12=16-4) \\
+  \quad 1101 \ (13=16-3) \\
\hline
11001 \rightarrow 25=16+16-7
\end{align*}
\]

Formula: \((2^n-x)+(2^n-y)=16-4+16-3=16+16-7\)

After we delete \(b_n\), the result is 16-7

Note that \(b_{n-1}=1\). Thus, the solution is negative.
4.3 Flow of 2’s Complement

- **Sub** = 0 for X+Y
- **Sub** = 1 for X-Y

**Inverter**
- Z = Y if Sub = 0
- 1’s comp of Y if Sub = 1

**Adder**
- Sum = A + B + Carry In
- Carry Out
- Carry In

**A** and **B**
- S = X + Y if Sub = 0
- S = X - Y (2’s comp) if Sub = 1

**b_n**
- Ignored
### 4.3 Addition and Subtraction: 1’s Comp.

<table>
<thead>
<tr>
<th>Arith.</th>
<th>Addition in 1’s</th>
<th>Sol. in 1’s comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + y$</td>
<td>$x + y$</td>
<td>$x + y$</td>
</tr>
</tbody>
</table>
| $x - y$ | $x + (2^n - 1 - y)$ | $2^n - 1 + (x - y)$ (if $x \leq y$)  
$x - y$ (if $x > y$) |
| $-x + y$ | $(2^n - 1 - x) + y$ | $2^n - 1 + (-x + y)$ (if $x \geq y$)  
$-x + y$ (if $x < y$) |
| $-x - y$ | $(2^n - 1 - x) + (2^n - 1 - y)$ | $2^n - 1 - x - y$ |
4.3 Addition and Subtraction: 1’s Comp.

Input: two positive integers x & y,
1. We represent the operands in one’s complement.
2. We sum up the two operands.
3. We delete $2^n - 1$ if $b_n = 1$.
4. The result is the solution in one’s complement.

<table>
<thead>
<tr>
<th>Arith.</th>
<th>Addition in 1’s</th>
<th>Result in 1’s comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>x - y</td>
<td>$x + (2^n - 1 - y) = 2^n - 1 + (x - y)$. $b_n = 1$ if $x &gt; y$</td>
<td>$2^n - 1 + (x - y)$ ($x \leq y$) $x - y$ ($x &gt; y$)</td>
</tr>
<tr>
<td>-x + y</td>
<td>$(2^n - 1 - x) + y = 2^n - 1 + (-x + y)$. $b_n = 1$ if $x &lt; y$</td>
<td>$2^n - 1 + (-x + y)$ ($x \geq y$) $-x + y$ ($x &lt; y$)</td>
</tr>
<tr>
<td>-x - y</td>
<td>$(2^n - 1 - x) + (2^n - 1 - y) = 2^n - 1 + 2^n - 1 - x - y$. $b_n = 1$</td>
<td>$2^n - 1 - x - y$</td>
</tr>
</tbody>
</table>
4.3 Addition and Subtraction: 1’s Comp.
Example: 4 – 3 = 1

In 1’s complement, we represent -3 as \((1100)_2\)

\[
\begin{array}{c}
\text{0100 (4)} \\
+ \text{1100 (12=15-3 )} \\
\text{10000 (16=15+1 )} \\
\text{0001(after deleting } 2^{n-1})
\end{array}
\]

Formula: \(x+(2^n-1-y)=4+(15-3)=15+1\)

We discard bit \(b_n (-2^n)\) and add one at the first bit (+1), i.e. deduct \(2^n-1\).
4.3 Addition and Subtraction: 1’s Comp.
Example: -4 +3 = -1

In 1’s complement, we represent -4 as (1011)_2

\[
\begin{align*}
1011 \ (11 &= 15 - 4) \\
+ 0011 \ (3) \\
\hline
1110 \ (14 &= 15 - 1)
\end{align*}
\]

Formula: \((2^n - 1 - x) + y = 15 - 4 + 3 = 14\)
Note that \(b_{n-1} = 1\). Thus, the solution is a negative number.
4.3 Addition and Subtraction: 1’s Comp.

Example: \(-4 - 3 = -7\)

In 1’s complement, we represent \(-4\) as \((1011)_2\)
\(-3\) as \((1100)_2\)

\[
\begin{align*}
1011 & \quad (11=15-4) \\
+ 1100 & \quad (12=15-3) \\
\hline
1,0111 & \quad (23=15+15-7)
\end{align*}
\]

So now take \(b_n=1\) and remove it from the 5th spot and add it to the remainder

\[
\begin{align*}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0111 \\
+ \quad 1 \\
\hline
1000 & \quad (8=15-7)
\end{align*}
\]
4.3 Flow of 1’s Complement

- **Sum**: $A + B + \text{Carry In}$
- **Carry Out**: $Z$
- **Adder**: $A$ (Carry Out) to $B$ (Carry In)
- **Inverter**: $X$ to $Y$
- **Sub**: $0$ for $X+Y$, $1$ for $X-Y$

- **Z**: $Y$ if $\text{Sub}=0$, $1’s$ comp of $Y$ if $\text{Sub}=1$
- **S**: $X+Y$ if $\text{Sub}=0$, $X-Y$ ($1’s$ comp) if $\text{Sub}=1$
4.3 Inverse Conversion

1's Compliment:
Let \( f(x) = 2^n - 1 - x \)
Theorem: \( f(f(x)) = x \)
Proof: \( f(f(x)) = f(2^n - 1 - x) \)
= \( f(2^n - 1 - x) \)
= \( 2^n - 1 - (2^n - 1 - x) \)
= \( x \)

2's Compliment:
Let \( g(x) = 2^n - x \)
Theorem: \( g(g(x)) = x \)
Proof: \( g(g(x)) = g(2^n - x) \)
= \( g(2^n - x) \)
= \( 2^n - (2^n - x) \)
= \( x \)
4.4 Overflow

Overflow occurs when the result lies beyond the range of the number system

Examples

Overflow Flag formula
### 4.4 Overflow: Examples (2’s Comp.)

2’ Comp: n=4, range -8 to 7

<table>
<thead>
<tr>
<th>Bit</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_a$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$X+Y$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bit</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
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<tbody>
<tr>
<td>$C_a$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>X</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$X+Y$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bit</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
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<tbody>
<tr>
<td>$C_a$</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>Y</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>$X+Y$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-5</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Bit</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
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<tbody>
<tr>
<td>$C_a$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>Y</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>$X+Y$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-9</td>
</tr>
</tbody>
</table>
4.4 Overflow: iCliker

For 2’s complement, overflow occurs when the following condition is true.

A. Both of $C_n$ and $C_{n-1}$ are one
B. Both of $C_n$ and $C_{n-1}$ are zero
C. Either of $C_n$ and $C_{n-1}$ is one but not both
D. None of the above.
4.4 Overflow Condition

Exercise:
1. State and prove the condition of the overflow of 1’s complement number system.
2. State and prove the condition of the overflow of 2’s complement number system.