1. (number systems: one’s complement) Show the operation of $17 + (-14)$ in one’s complement of binary number system. Assume that each binary number is represented with 10 bits. (10 points)

$$
17 = 10001_2 \\
= 000010001_2 \\
14 = 1110_2 \\
= 000001110_2 \\
-14 = \underbrace{1111111111_2} - \underbrace{0000011110_2} \\
= 111110001_2 \\
17 - 14 = 0000010012 + 1111100012 \\
= 0000000102 + 1 \\
= 000000011_2
$$

2. (number systems: two’s complement) We have defined and learned the idea of two’s complement for n-bit binary numbers.
2.1. Define the complement (corresponding to two’s) using an n-digit system with base 8. (5 points)

$$
-x \Rightarrow 8^n - x \\
Range = [-4 \times 8^{n-1}, 4 \times 8^{n-1} - 1]
$$

2.2. Show the arithmetic of $x - y$ where $x = 11_8$ and $y = 17_8$ in the complement representations (corresponding to two’s) using a 5-digit system with base 8. (5 points)

$$
x = 11_8 \\
= 00011_8 \\
y = 00017_8 \\
-y = 77777_8 - 00017_8 + 1_8 \\
= 77761_8 \\
x - y = 00011_8 + 77761_8 \\
= 77772_8
$$
3. (Boolean algebra: proof of consensus theorem) Prove the following equality using Boolean algebra laws and theorems.

3.1 Prove the consensus theorem: \( ab + a'c = ab + a'c + bc. \) (5 points)

\[
\begin{align*}
ab + a'c + bc &= ab + a'c + 1 \times bc \\
&= ab + a'c + (a + a')bc \\
&= ab + a'c + abc + a'bc \\
&= ab + abc + a'c + a'bc \\
&= ab(1 + c) + a'c(1 + b) \\
&= ab + a'c
\end{align*}
\]

3.2 Prove the Boolean equality \((a + b)(a' + c) = (a + b)(a' + c)(b + c).\) (5 points)

\[
\begin{align*}
(a + b)(a' + c)(b + c) &= (aa' + ac + a'b + bc)(b + c) \\
&= (ac + a'b + bc)(b + c) \\
&= abc + acc + a'bb + a'bc + bbc + bcc \\
&= abc + ac + a'b + a'bc + bc \\
&= ac(b + 1) + a'b(c + 1) + bc \\
&= ac + a'b + bc \\
&= 0 + ac + a'b + bc \\
&= a'a + ac + a'b + bc \\
&= a(a' + c) + b(a' + c) \\
&= (a + b)(a' + c)
\end{align*}
\]
4. (Boolean algebra) Express Boolean function \( E(a, b, c) = (b + ac')'(ab' + c) \) in sum-of-products form using Boolean algebra laws and theorems. Express in the minimal expression. (10 points)

\[
E(a, b, c) = (b + ac')'(ab' + c) \\
= (b')(ac')'(ab' + c) \quad \text{(DeMorgan)} \\
= b'(a' + c)(ab' + c) \quad \text{(distributive)} \\
= (a'b' + b'c)(ab' + c) \quad \text{(distributive)} \\
= ab'ab' + a'b'c + ab'b'c + b'cc \quad \text{(complement)} \quad (6) \\
= a'b'c + ab'b'c + b'cc \quad \text{(idempotent)} \\
= a'b'c + ab'c + b'c \quad \text{(distributive)} \\
= (a' + a + 1)b'c \quad \text{(complement)} \\
= b'c
\]

5. (Boolean algebra: product of sums) Express Boolean function \( E(a, b, c) = a'bc + b'[(a + b')(a + c)]' \) in product-of-sums form using Boolean algebra laws and theorems. Express in the minimal expression. (10 points)

\[
E(a, b, c) = a'bc + b'[(a + b')(a + c)]' \\
= a'bc + b'[(a + b')' + (a + c)]' \quad \text{(DeMorgan)} \\
= a'bc + b'(a'b + d'c') \quad \text{(distributive)} \\
= a'bc + a'bb' + a'b'c' \quad \text{(complement)} \quad (7) \\
= a'bc + a'b'c' \quad \text{(distributive)} \\
= a'(bc + b'c') \quad \text{(distributive)} \\
= a'(b + b'c')(c + b'c') \quad \text{(theorem8)} \\
= a'(b + c')(b' + c)
\]

3
6. (recursive function: permutation) Suppose all the permutations on the set of \{1, 2, 3, 4, 5, 6\} are listed in lexicographic order from 0 to 6! – 1.

6.1 What is the RANK (order) in the list for 453261? (10 points)

\[
RANK(453261) = 3 \times 5! + 3 \times 4! + 2 \times 3! + 1 \times 2! + 1 \times 1!
= 3 \times 120 + 3 \times 24 + 2 \times 6 + 1 \times 2 + 1 \times 1
= 360 + 72 + 12 + 2 + 1
= 447
\]

6.2 What permutation will have the RANK 165? (5 points)

Assume the permutation to be \(a_6a_5a_4a_3a_2a_1\), local rank to be \(r_i\) for \(i\)th level. Originally we start from level 6 and have \(r_6 = 165\).

\[
\begin{align*}
q_6 &= r_6/5! = 1 \\
r_5 &= r_6 - q_6 \times 5! = 45 \\
a_6 &= \{1, 2, 3, 4, 5, 6\} = 2 \\
q_5 &= r_5/4! = 1 \\
a_5 &= \{1, 3, 4, 5, 6\} = 3 \\
r_4 &= r_5 - q_5 \times 4! = 21 \\
q_4 &= r_4/3! = 3 \\
a_4 &= \{1, 4, 5, 6\} = 6 \\
r_3 &= r_4 - q_4 \times 3! = 3 \\
q_3 &= r_3/2! = 1 \\
a_3 &= \{1, 4, 5\} = 4 \\
r_2 &= r_3 - q_3 \times 2! = 1 \\
q_2 &= r_2/1! = 1 \\
a_2 &= \{1, 5\} = 5 \\
r_1 &= r_2 - q_2 \times 1! = 0 \\
q_1 &= r_1/0! = 0 \\
a_1 &= \{1\} = 1
\end{align*}
\]

So we have the permutation for RANK 165 as

\(a_6a_5a_4a_3a_2a_1 = 236451\)
7. (recursive function: induction) Use induction to prove the following identity for any positive integer \( n \): 
\[
1 \times 2 + 2 \times 3 + \ldots + (n - 1) \times n = n(n - 1)(n + 1)/3.
\]
(10 points)

From the question we have the following.

\[
a_n = n(n - 1)
\sum(n) = \sum_{i=1}^{n} a_i
f(n) = n(n - 1)(n + 1)/3
\]

And we need to prove that

\[\forall n \in \{1, 2, \ldots, +\infty\}, sum(n) = f(n)\]

Base case:

\[
sum(1) = 1 \times (1 - 1) = 0
f(1) = 1 \times (1 - 1) \times (1 + 1)/3 = 0
\Rightarrow sum(1) = f(1)
\]

Assumption:

\[\exists k > 0 \text{ s.t. } sum(k) = \sum_{i=1}^{k} a_i = f(k) = k(k - 1)(k + 1)/3\]

Incremental case:

\[
sum(k + 1) = \sum_{i=1}^{k+1} a_i = \sum_{i=1}^{k} a_i + a_{k+1}
= sum(k) + a_{k+1}
= k(k - 1)(k + 1)/3 + k(k + 1)
= k(k + 1)((k - 1)/3 + 1)
= k(k + 1)(k + 2)/3
= [((k + 1) - 1][k + 1] + 1]/3
= f(k + 1)
\]
Define the combination of 3 cents and 5 cents for \( n \) cents as

\[
 n = c_3(n) \times 3 + c_5(n) \times 5
\]

where \( c_3(n) \) and \( c_5(n) \) are positive integers or zero.

Define the feasibility as a boolean variable \( a_n \).

\[
a_n = \text{true} \iff \exists c_3(n) \& c_5(n) \text{ s.t. } n = c_3(n) \times 3 + c_5(n) \times 5
\]

We want to prove

\[
a_n = \text{true} \ \forall n \geq 8
\]

Base case:

\[
\begin{align*}
8 &= 1 \times 3 + 1 \times 5 \implies c_3(8) = 1 & & c_5(8) = 1 \implies a_8 = \text{true} \\
9 &= 3 \times 3 + 0 \times 5 \implies c_3(9) = 3 & & c_5(9) = 0 \implies a_9 = \text{true} \\
10 &= 0 \times 3 + 2 \times 5 \implies c_3(10) = 0 & & c_5(10) = 2 \implies a_{10} = \text{true}
\end{align*}
\]

Assumption:

\[
\forall n \in [8, k], \exists c_3(n) \& c_5(n) \text{ s.t. } n = c_3(n) \times 3 + c_5(n) \times 5 \& a_n = \text{true}
\]

Incremental case:

\[
k + 1 = (k + 1 - 3) + 3 \\
= (k - 2) + 3 \\
= c_3(k - 2) \times 3 + c_5(k - 2) \times 5 + 3 \\
= (c_3(k - 2) + 1) \times 3 + c_5(k - 2) \times 5
\]

\[
\implies \exists c_3(k + 1) = c_3(k - 2) + 1 \\
c_5(k + 1) = c_5(k - 2) \\
\implies a_{k+1} = \text{true}
\]
9. (recursive function) A frog knows 3 jumping styles \((A, B, C)\). With style \(A\) the frog jumps forward by 1 feet, and with styles, \(B, C\), the frog jumps forward by 2 feet. Let \(a_i\) denote the number of ways to jump over a total distance of \(i\) feet.

9.1. Write the values of \(a_1, a_2, a_3\)? (5 points)

\[
\begin{align*}
a_1 &= 1 \\
a_2 &= 1 \times 1 + 2 = 3 \\
a_3 &= 1 \times a_2 + 2 \times a_1 = 5
\end{align*}
\] (15)

9.2. Derive the recursive formula of \(a_n\)? (5 points)

\[a_n = a_{n-1} + 2 \times a_{n-2}\]

9.3. Find the solution of the recursion. (5 points)

\[
x^2 = x + 2
\]
\[
= x^2 - x - 2
\]
\[
= (x-2)(x+1)
\]
\[\Rightarrow r_1 = 2\]
\[r_2 = -1\]
\[\Rightarrow a_n = c_1 (r_1)^n + c_2 (r_2)^n\]
\[= c_1 2^n + c_2 (-1)^n\]
\[a_1 = 2c_1 - c_2 = 1\]
\[a_2 = 4c_1 + c_2 = 3\]
\[\Rightarrow c_1 = \frac{2}{3}\]
\[c_2 = \frac{1}{3}\]
\[\Rightarrow a_n = \frac{2}{3} \times 2^n + \frac{1}{3} \times (-1)^n\] (16)