CSE 202: Design and Analysis of Algorithms

Lecture 9

Instructor: Kamalika Chaudhuri

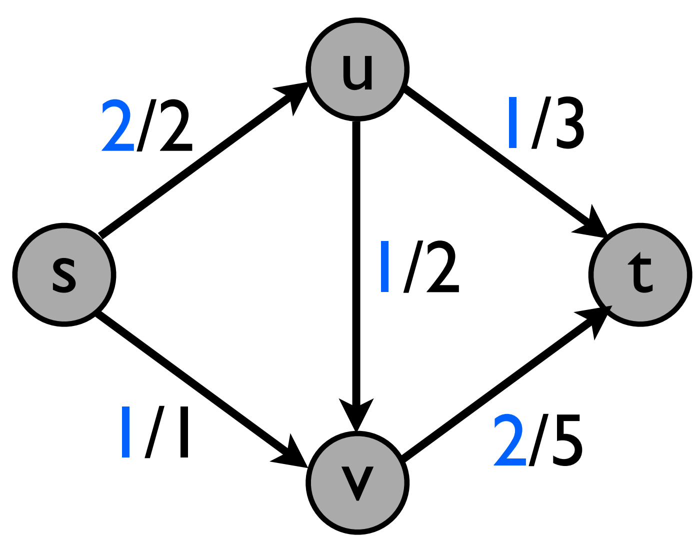
Last Class: Max Flow Problem

The Max Flow Problem: Given directed graph G=(V,E), source s, sink t, edge capacities c(e), find an s-t flow of maximum size

An s-t flow is a function f: E $\longrightarrow$ R such that:
- 0 <= f(e) <= c(e), for all edges e
- flow into node v = flow out of node v, for all nodes v except s and t,

$$\sum_{e\ into\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$$

Size of flow f = Total flow out of s = total flow into t



Size of f = 3

Last Class: Facts about Flows

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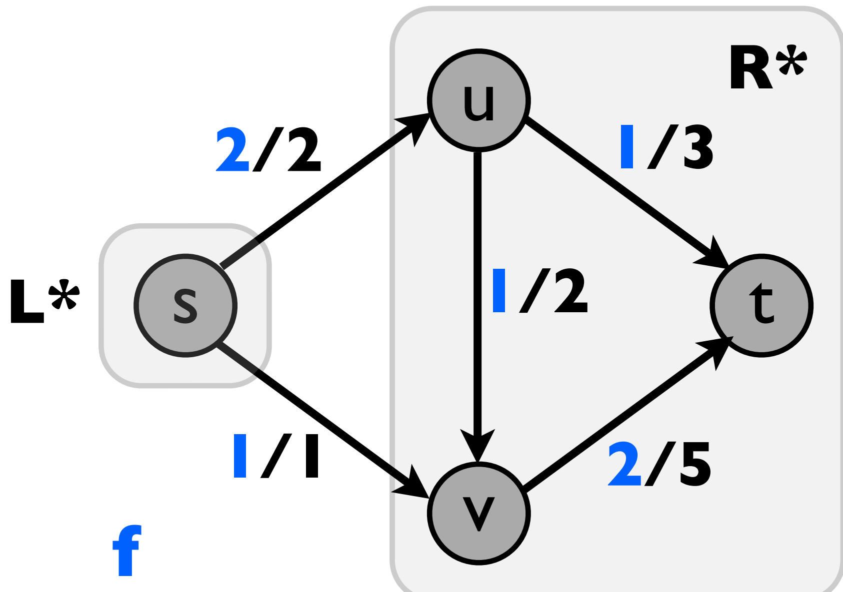


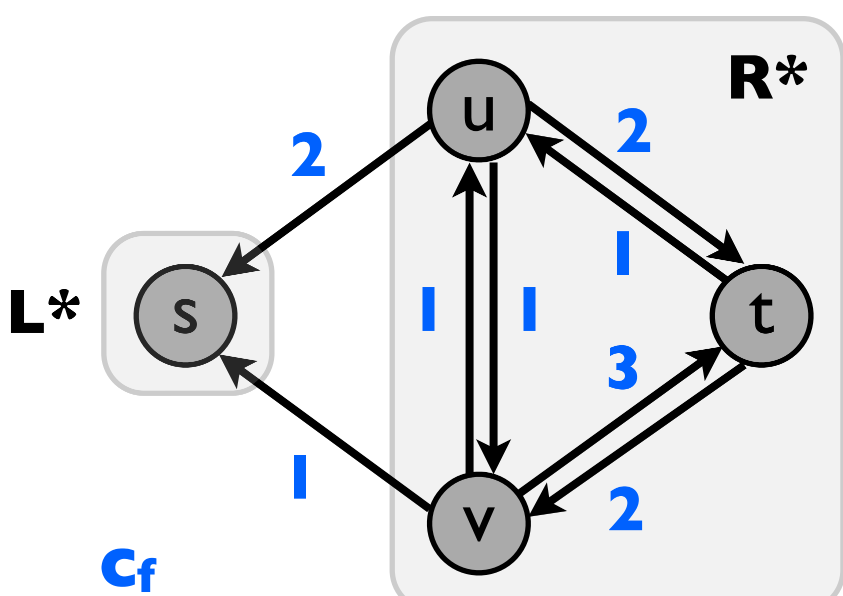
The Residual Graph: For a flow f

$G_f = (V, E_f)$ where $E_f \subseteq E \cup E^R$

For any (u,v) in E or E^R, **residual capacity:**

$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

[ignore edges with zero c_f: don't put them in E_f]



Max Flow Min Cut Theorem:

Size(Max-Flow) = Capacity(Min-Cut)

When is f a max flow?

When t is not reachable from s in G_f

Last Class: Algorithms for Max-Flow

Recall: n = #vertices, m = #edges in G

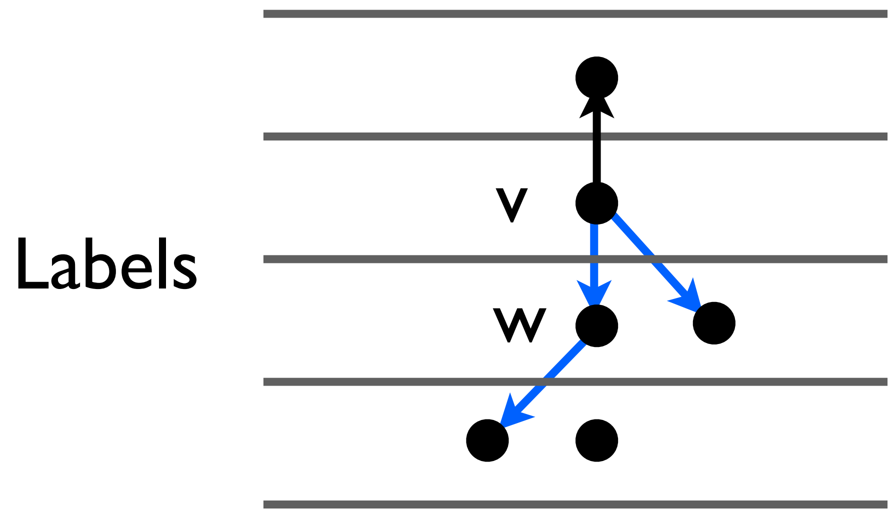
- Ford-Fulkerson: Running Time = $O(m\, C_{max})$

- Other efficient Ford-Fulkerson Style Algorithms:
  - Edmonds-Karp: Running Time = $O(nm^2)$
  - Capacity Scaling: Running Time = $O(m^2 \log C_{max})$

- Preflow-Push

Preflow-Push

Main Idea:
- Each node has a label, which is a potential
- Route flow from high to low potential



Idea: Route flow along blue edges

Preflows

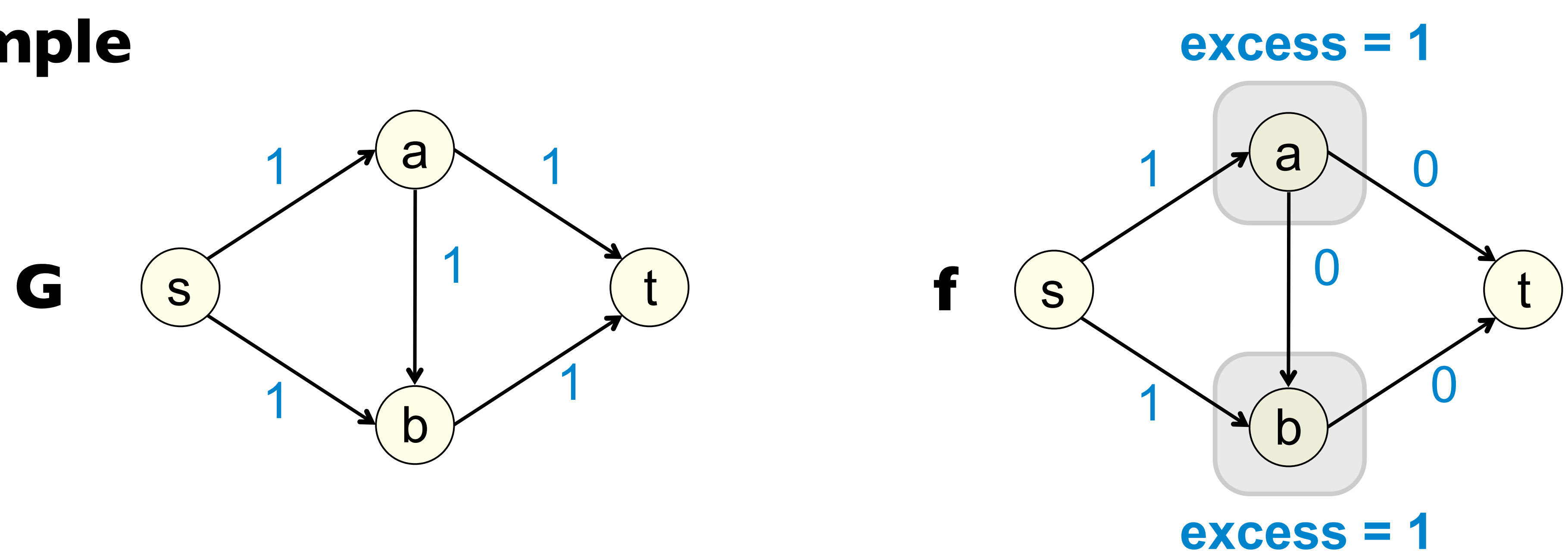
Preflow: A function f: E$\rightarrow$R is a preflow if:

1. **Capacity Constraints:** 0 <= f(e) <= c(e)
2. Instead of conservation constraints:

$$\sum_{e\ into\ v} f(e) - \sum_{e\ out\ of\ v} f(e) \geq 0$$

Excess(v) = $\displaystyle\sum_{e\ into\ v} f(e) - \sum_{e\ out\ of\ v} f(e)$

Example



Preflow-Push: Two Operations

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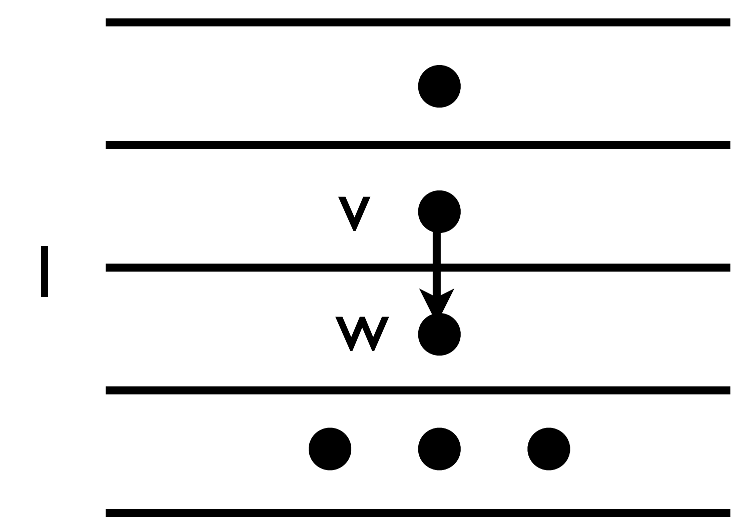
Labeling h assigns a non-negative integer label h(v) to all v in V



Push(v, w): Applies if excess(v) > 0, h(w) < h(v), (v, w) in E_f
  q = min(excess(v), c_f(v,w))
  Add q to f(v, w)

Relabel(v): Applies if excess(v) > 0, for all w s.t (v, w) in E_f, h(w) >= h(v)
  Increase h(v) by 1

Pre-Flow Push: The Algorithm

Start with labeling: $h(s) = n$, $h(t) = 0$, $h(v) = 0$, for all other v
Start with preflow f: $f(e) = c(e)$ for $e = (s, v)$, $f(e) = 0$, for all other edges e

While there is a node (other than t) with positive excess
  Pick a node v with excess(v) > 0
  If there is an edge (v, w) in E_f such that push(v, w) can be applied
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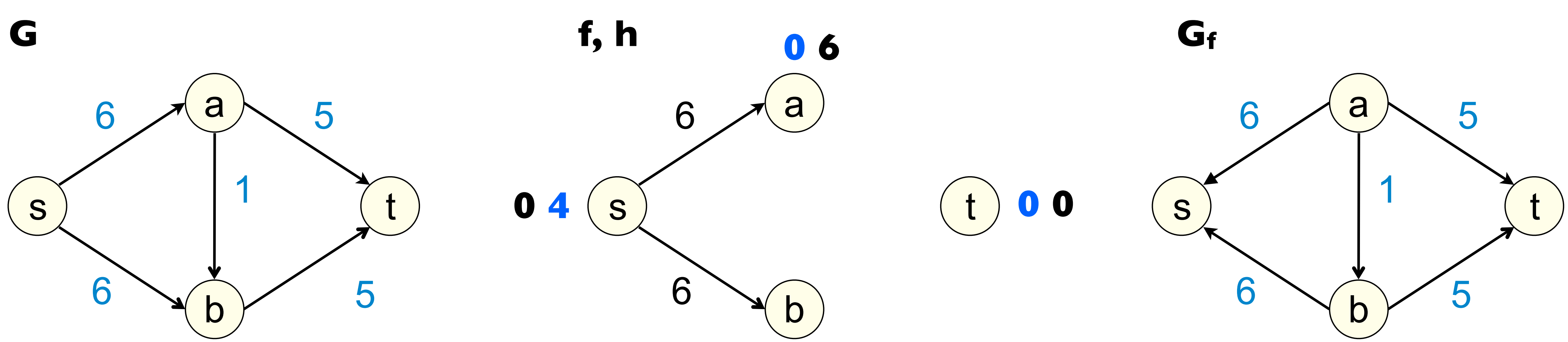

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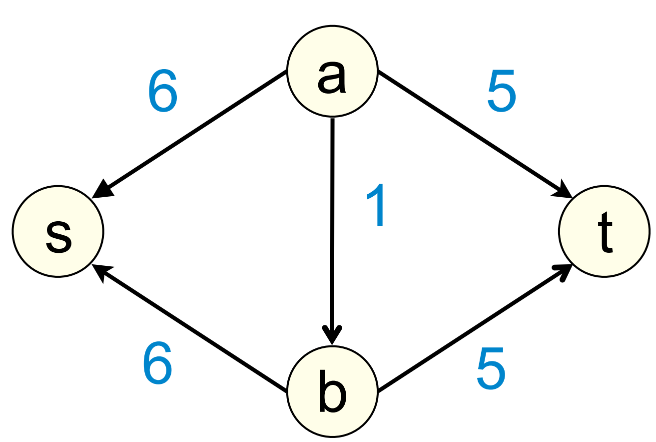

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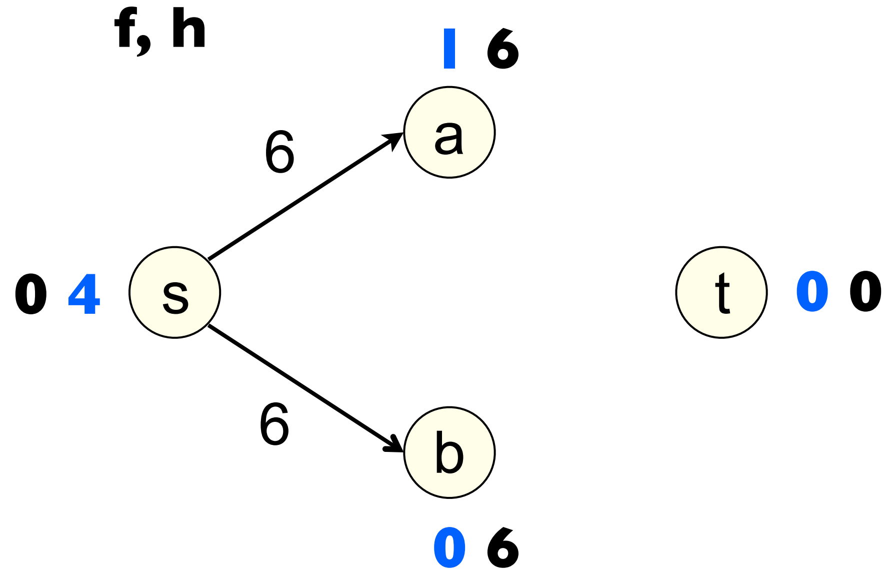

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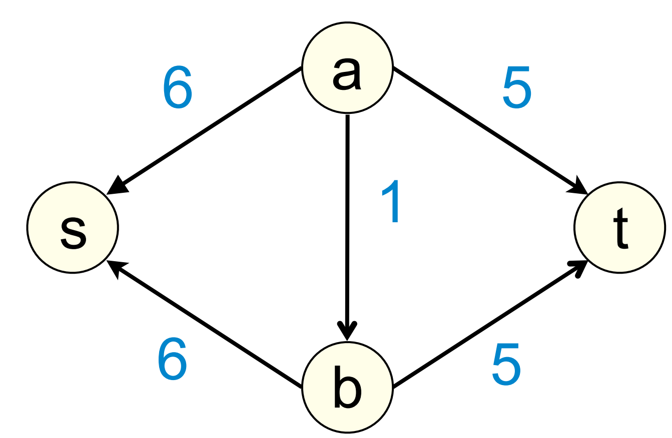

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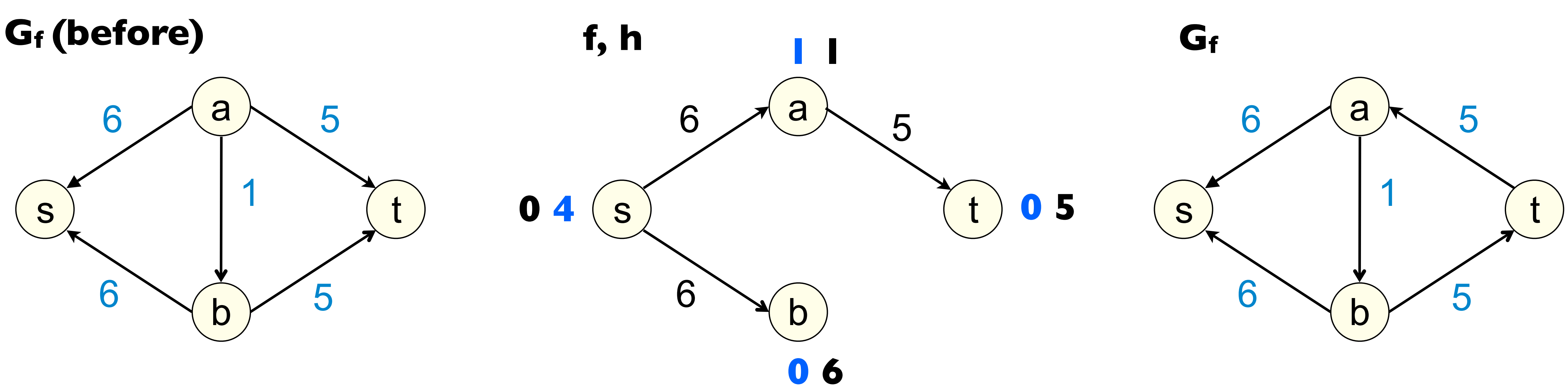

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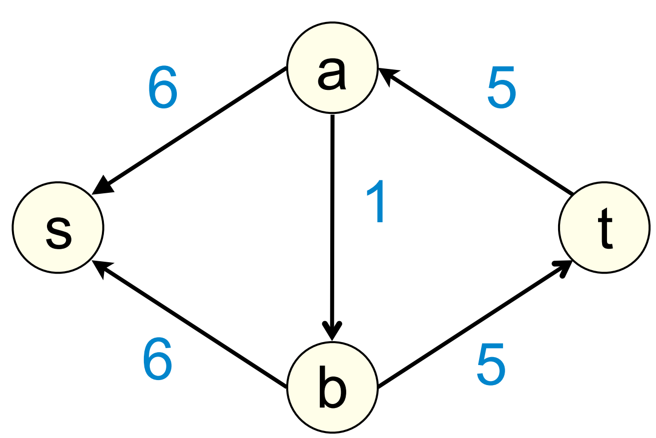

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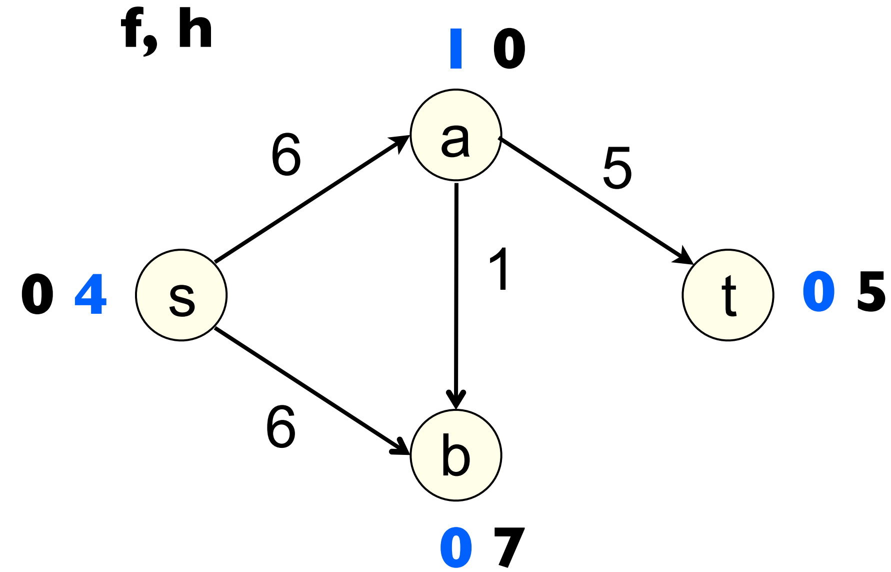

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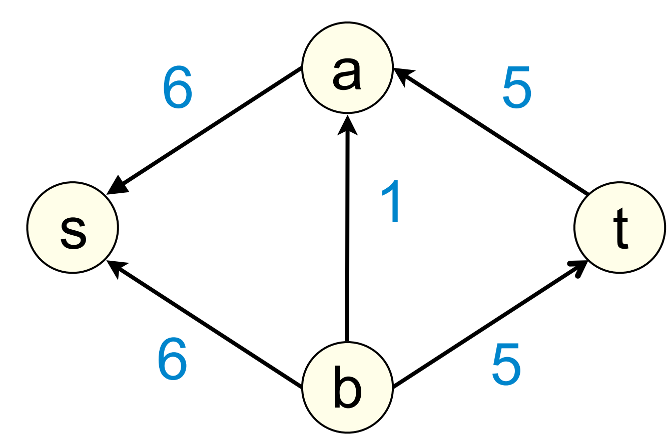

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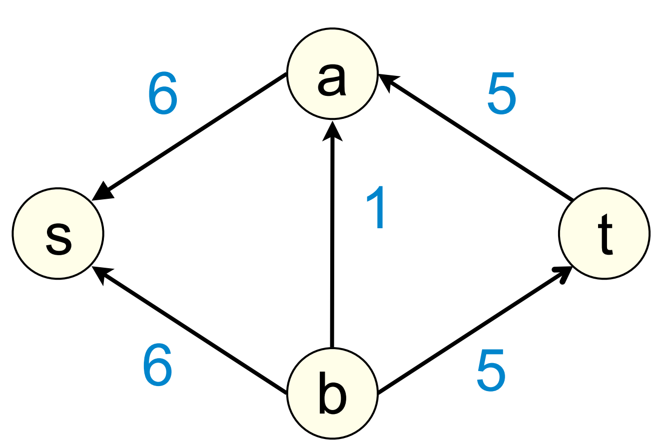

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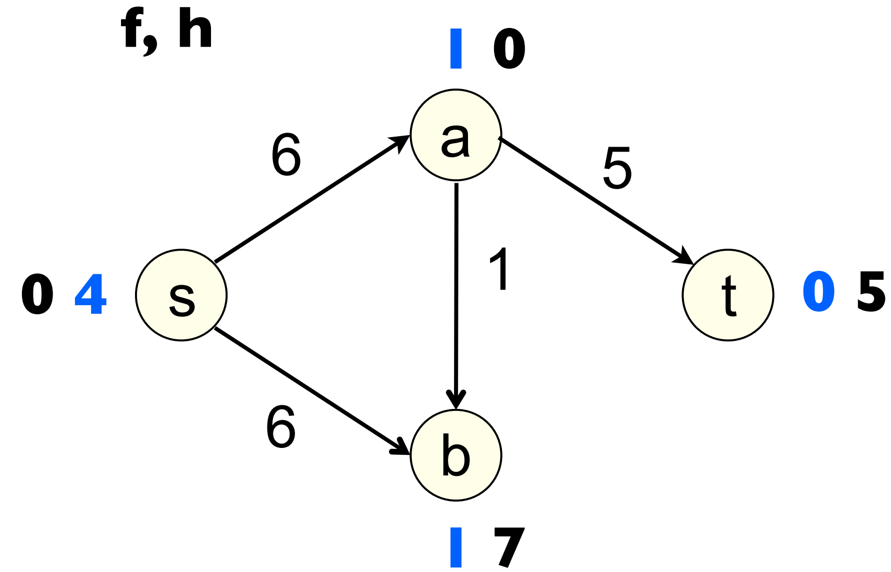

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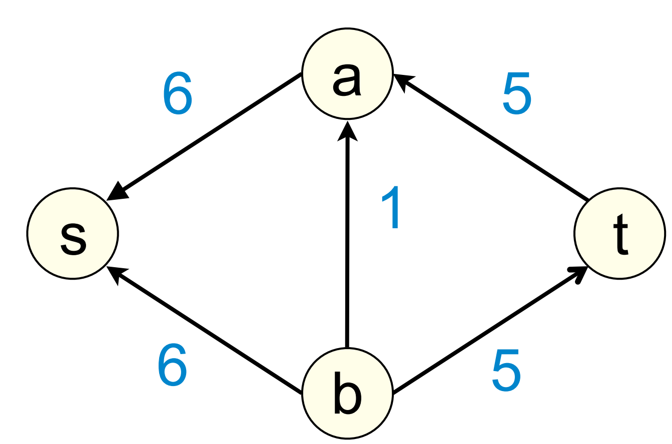

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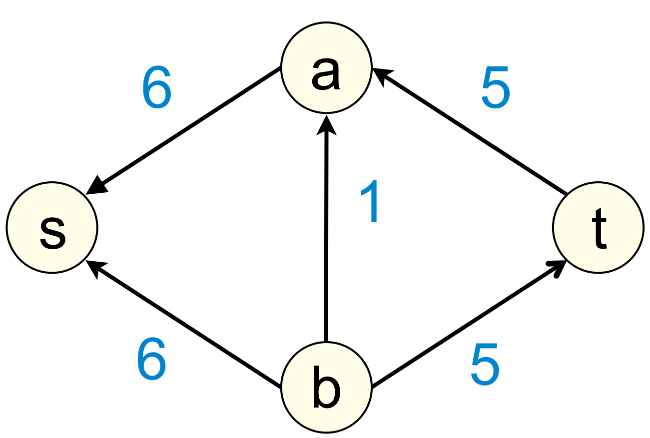

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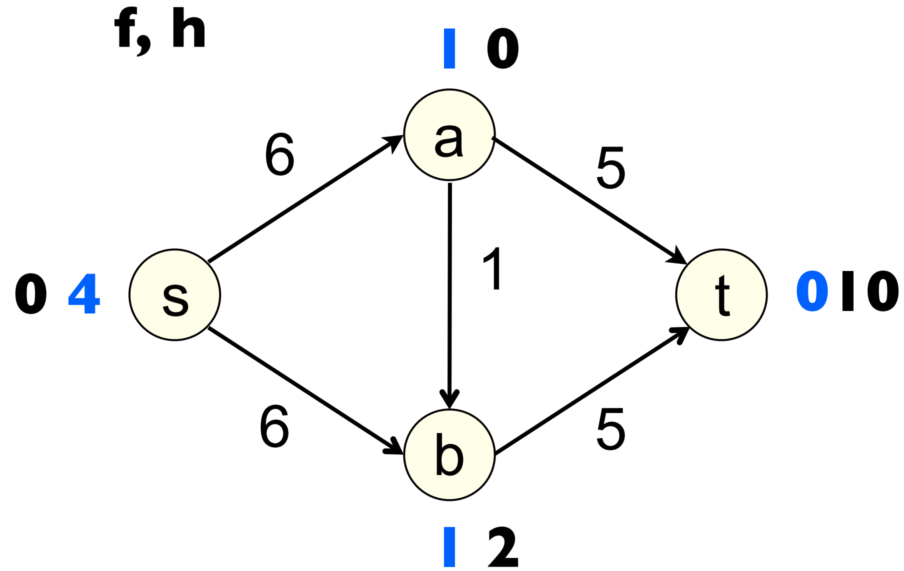

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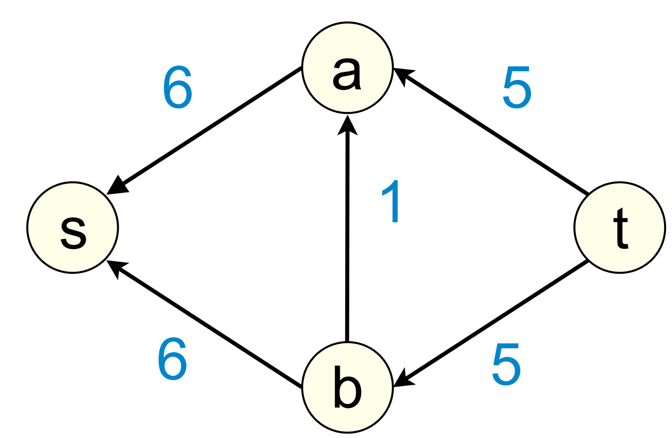

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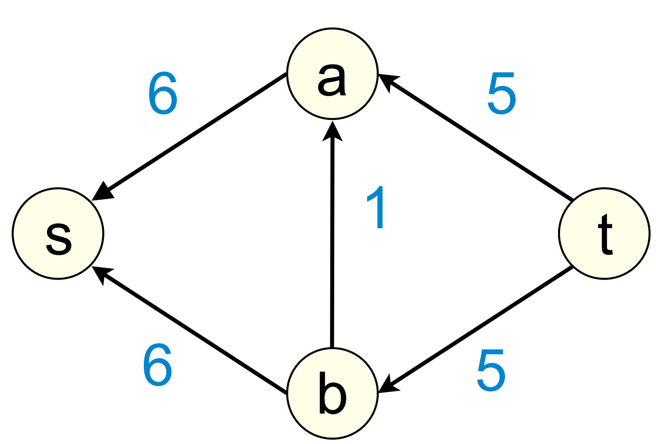

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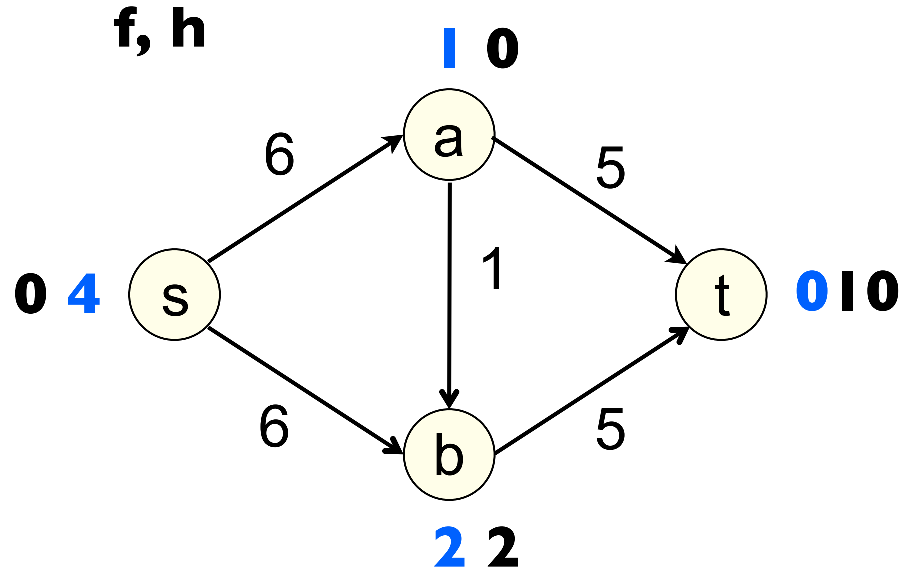

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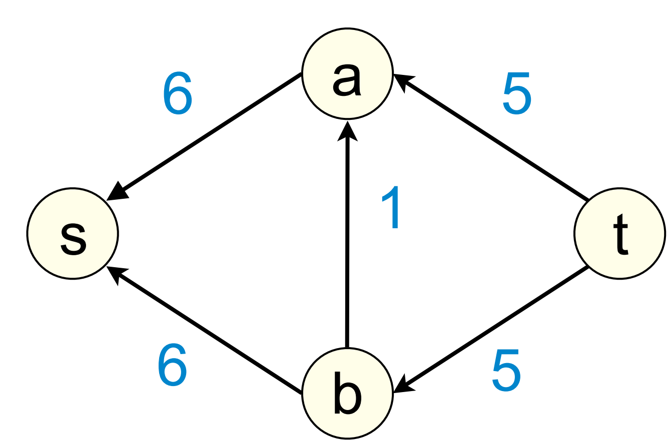

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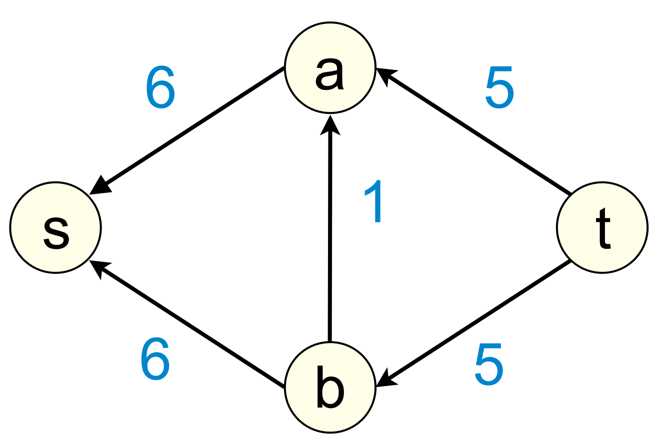

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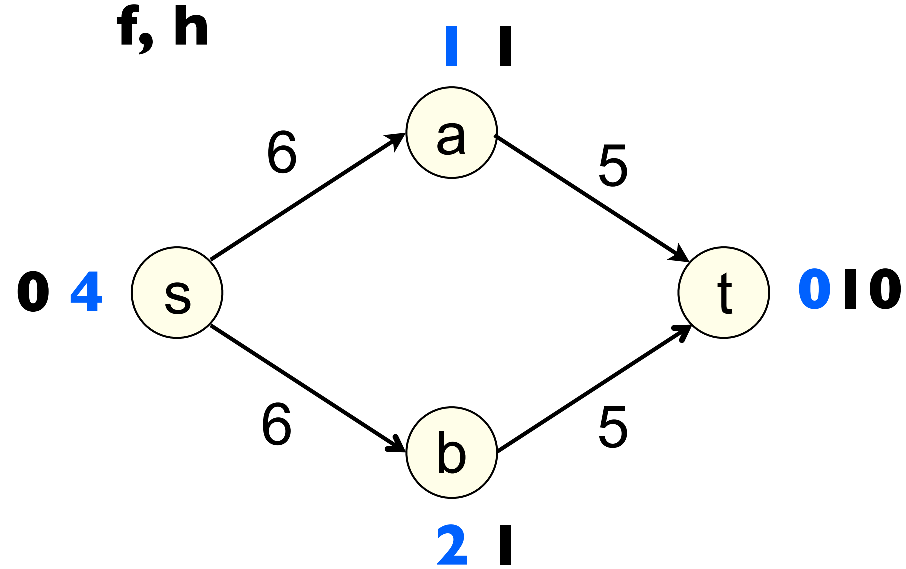

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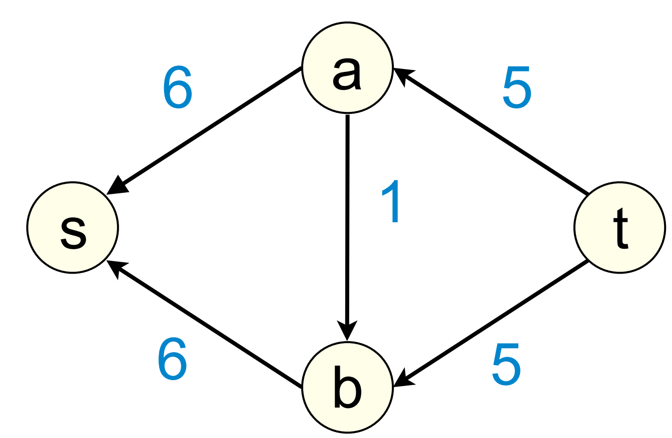

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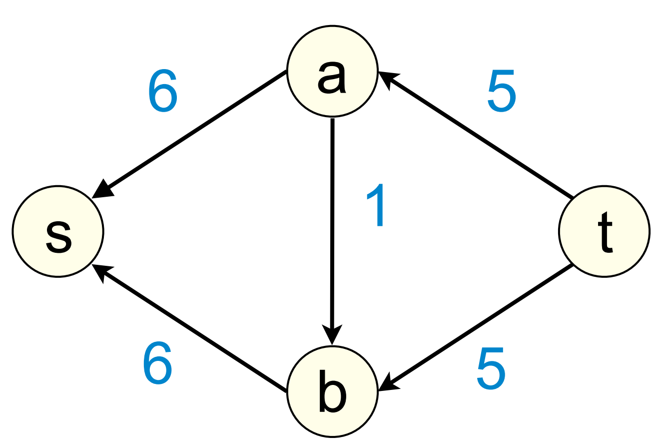

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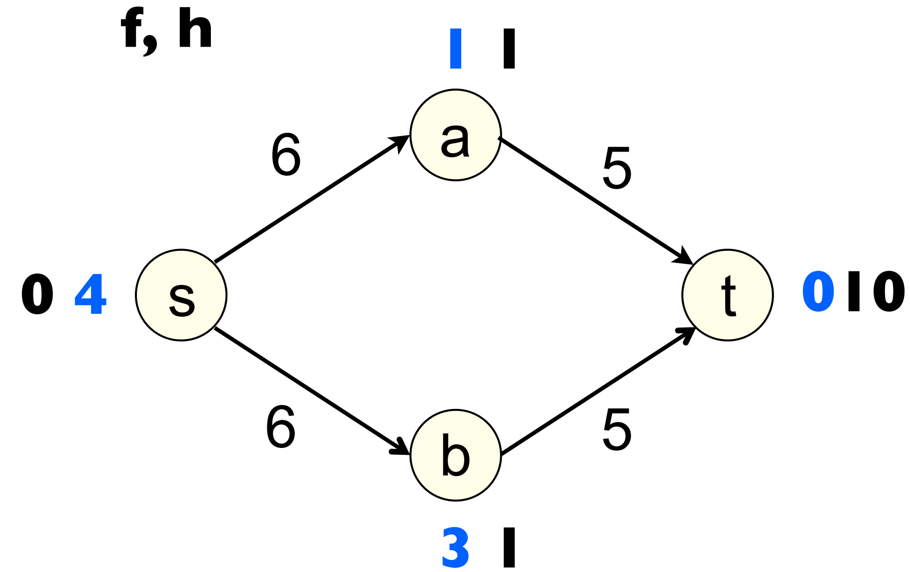

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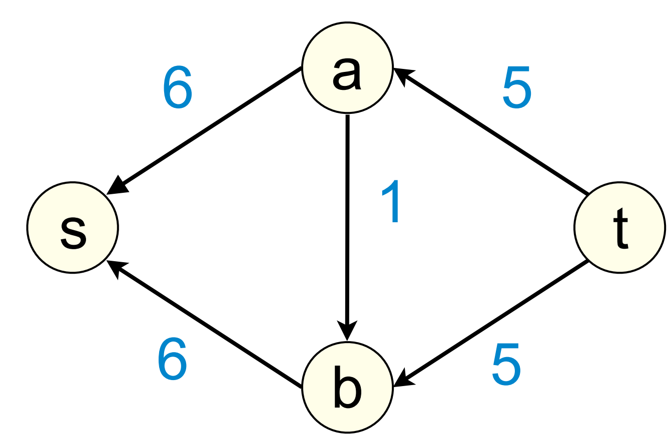

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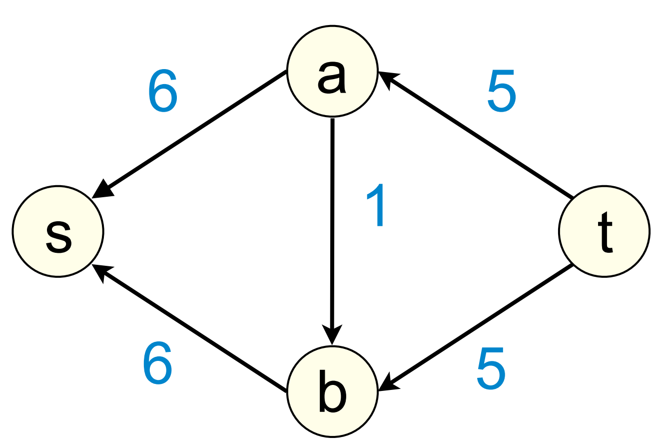

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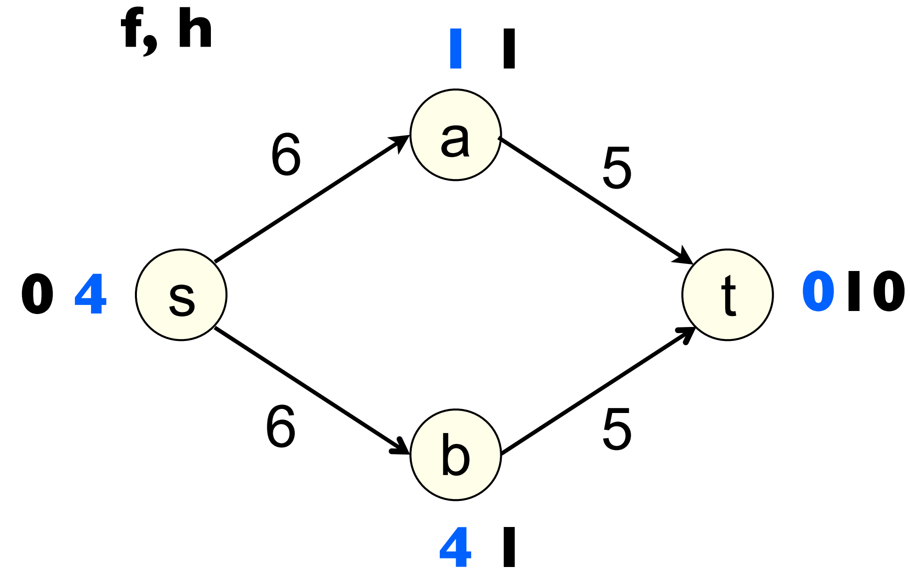

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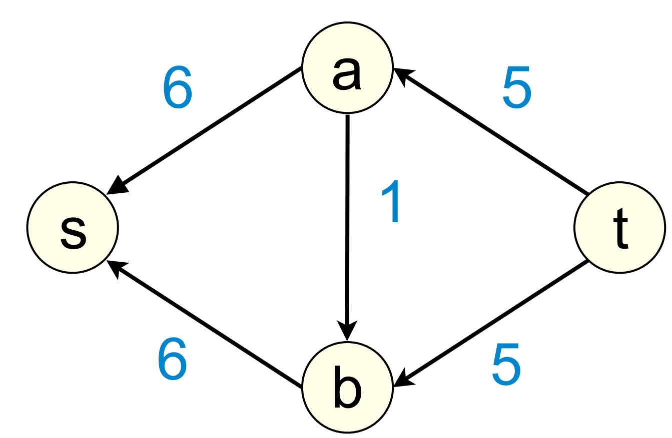

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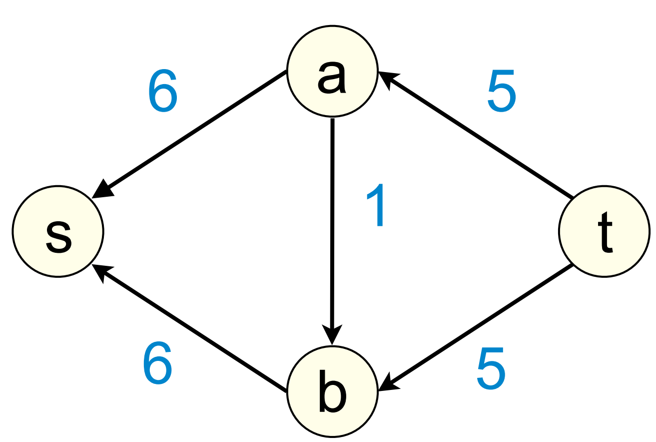

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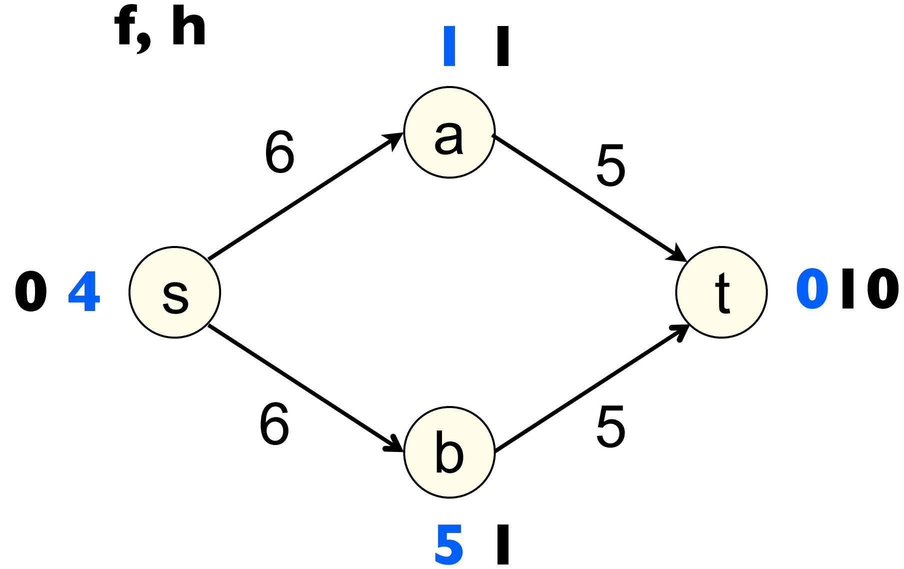

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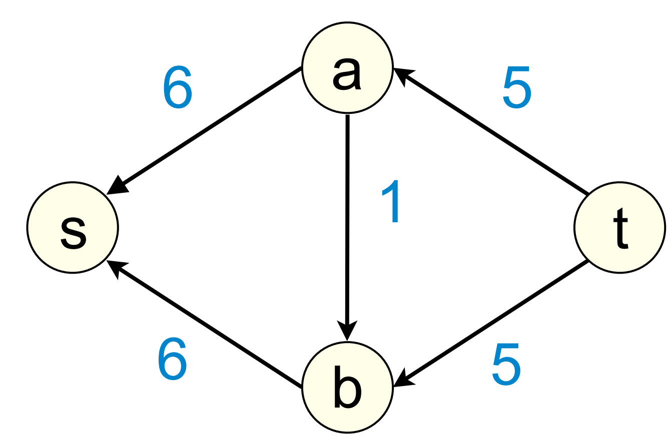

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- Pick a node v with excess(v) > 0
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Push(v, w):
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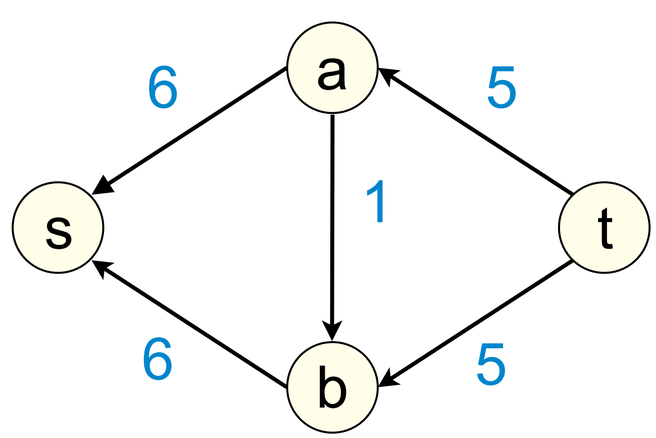
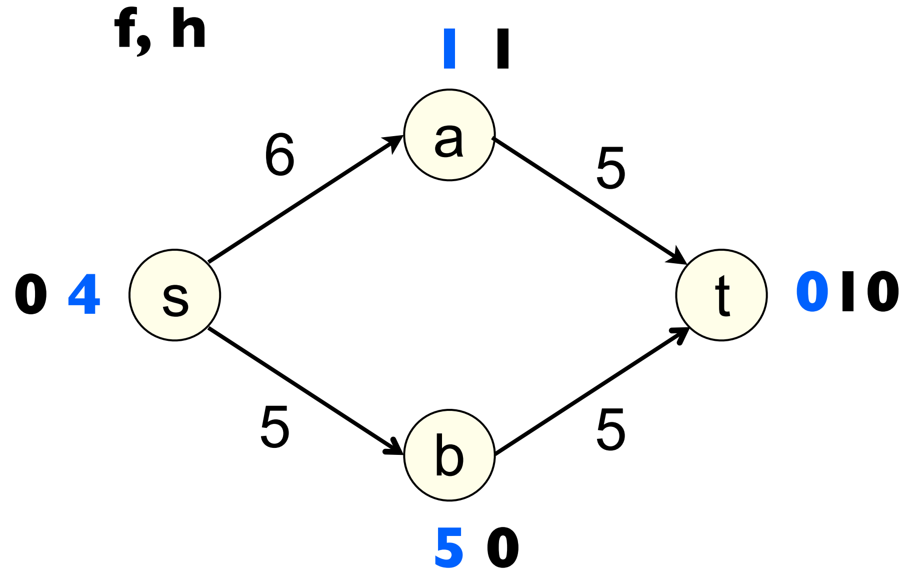
G_f (before)



f, h



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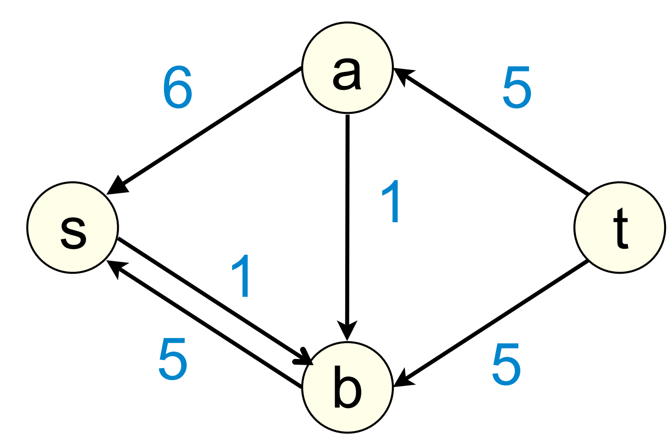

Labels

Excesses

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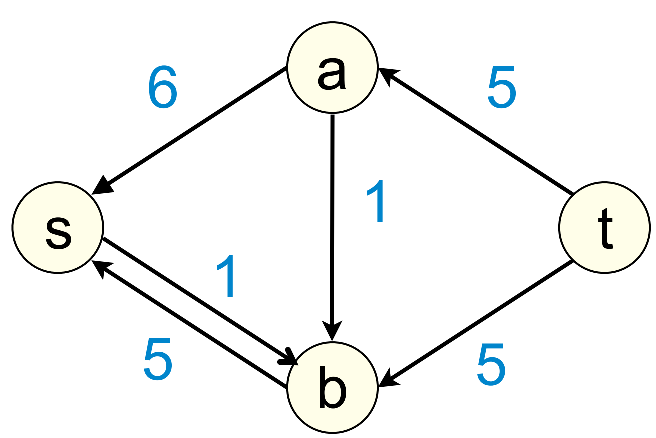

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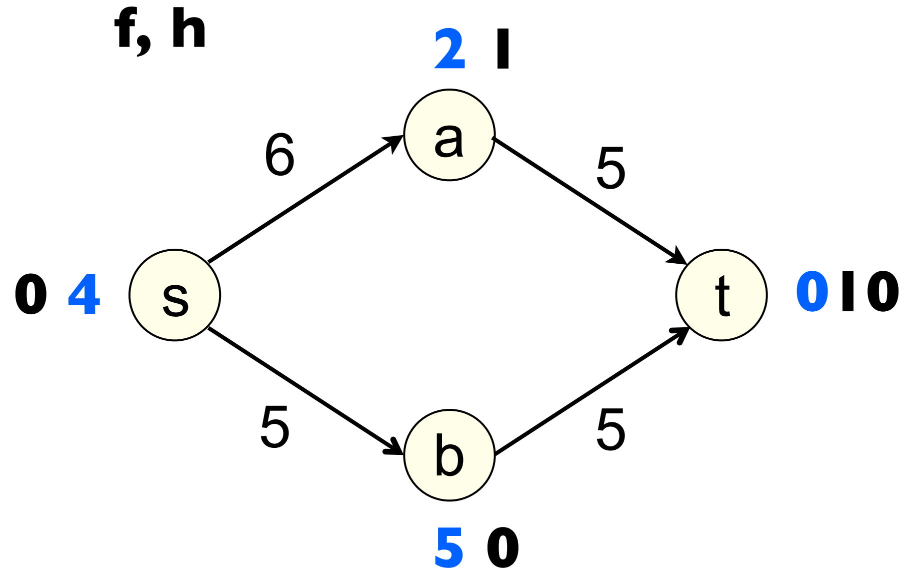

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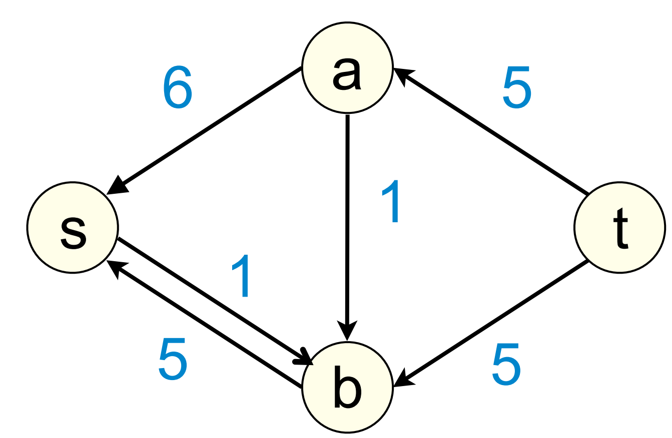

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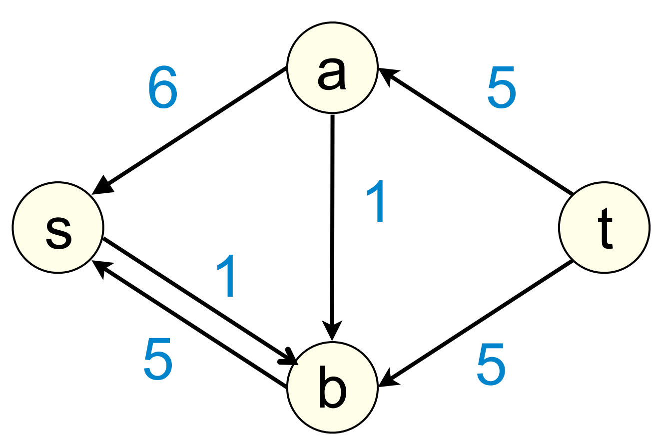

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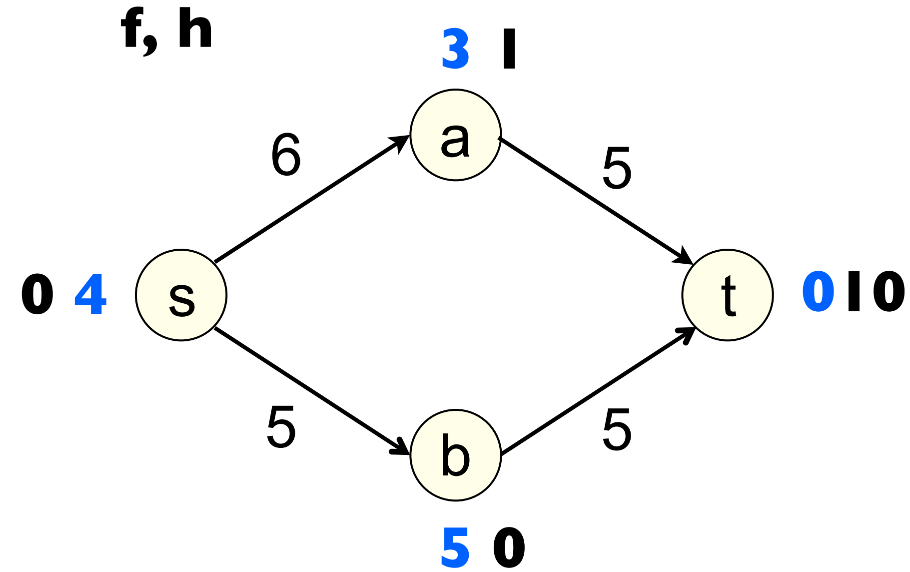

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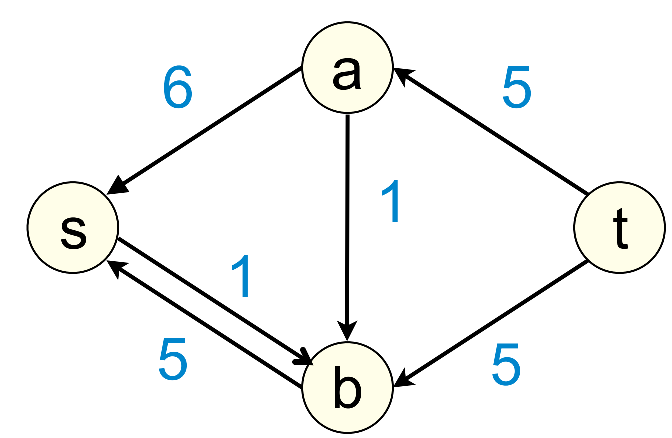

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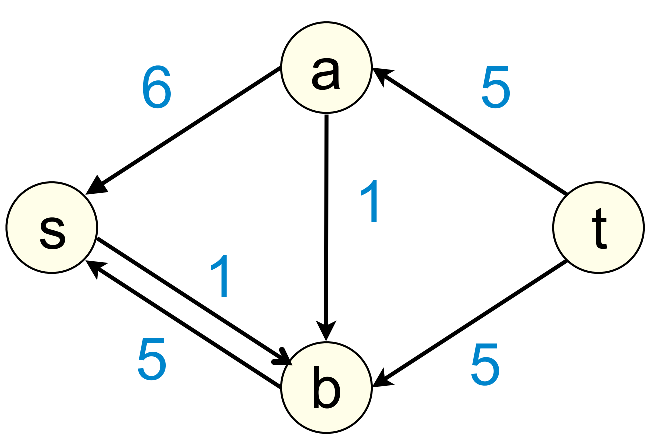

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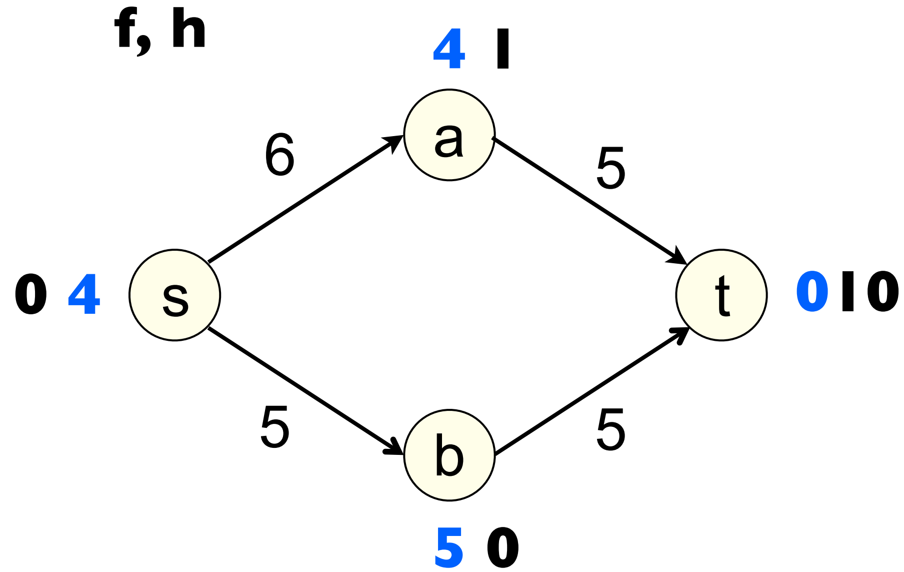

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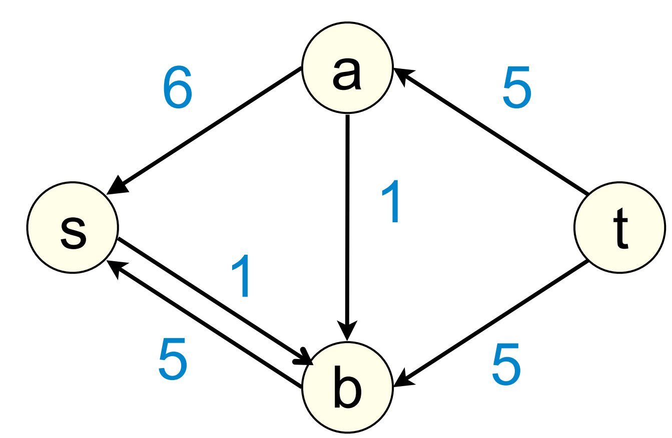

Labels

Excesses

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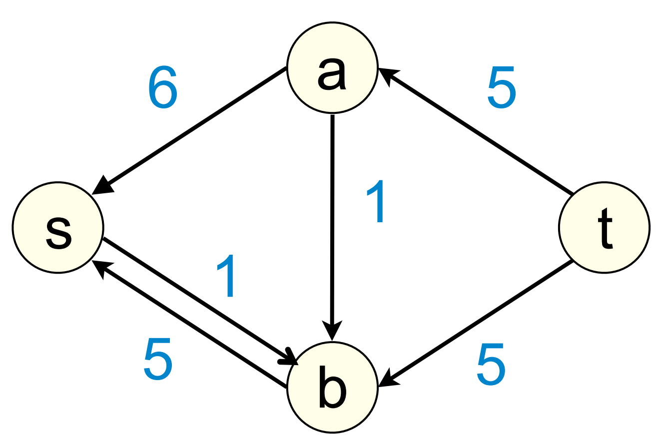

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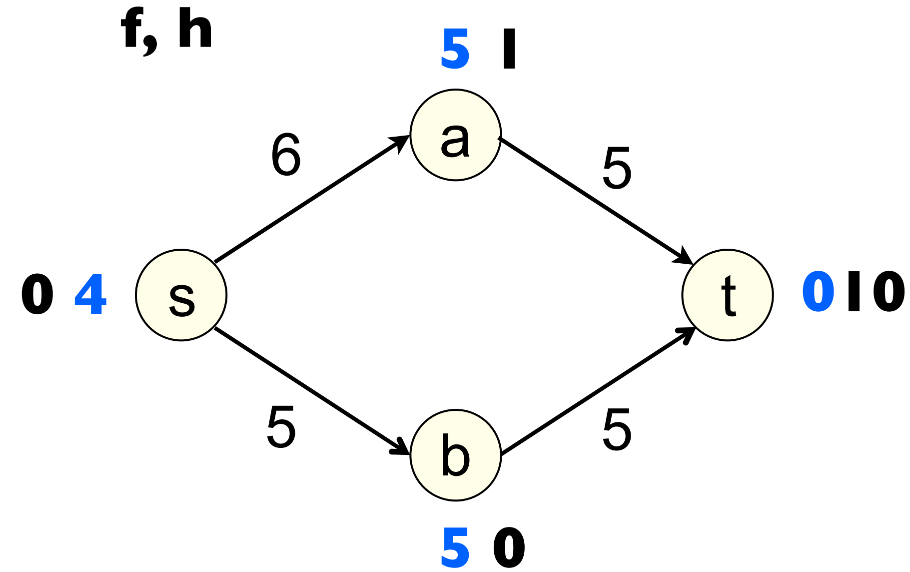
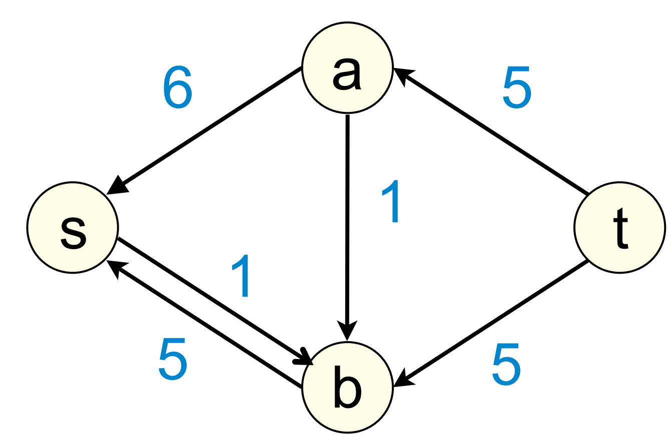


G_f



Labels

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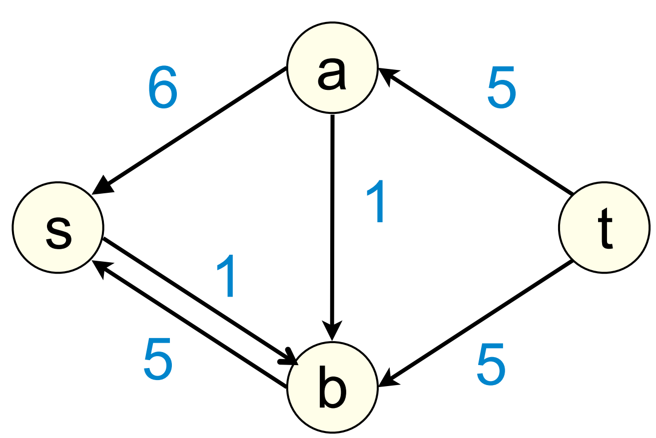

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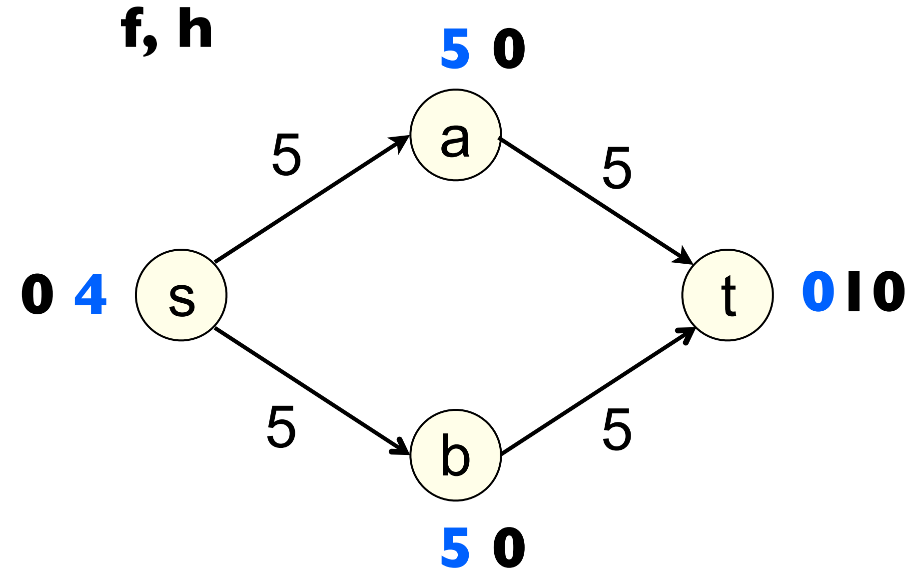

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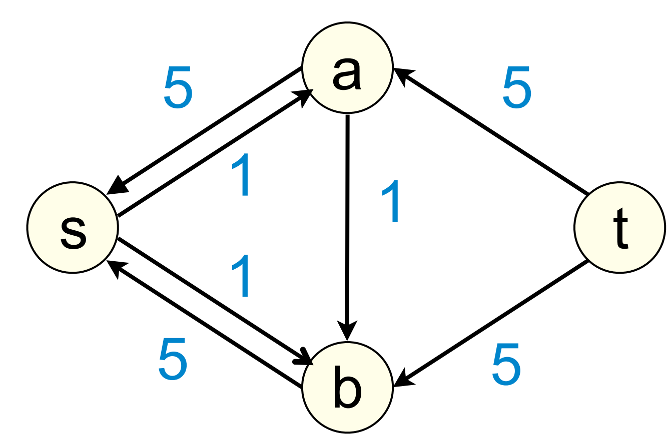


Labels

Excesses

Pre-Flow Push

- Algorithm
- Correctness
- Running Time Analysis

Correctness: Proof Outline

Three Steps:

- Compatibility: Show that the preflow f and the labeling h maintained by the algorithm always obeys a **compatibility** property

- If a flow f is compatible with some labeling, then f is a max-flow

- Preflow-push outputs a flow on termination

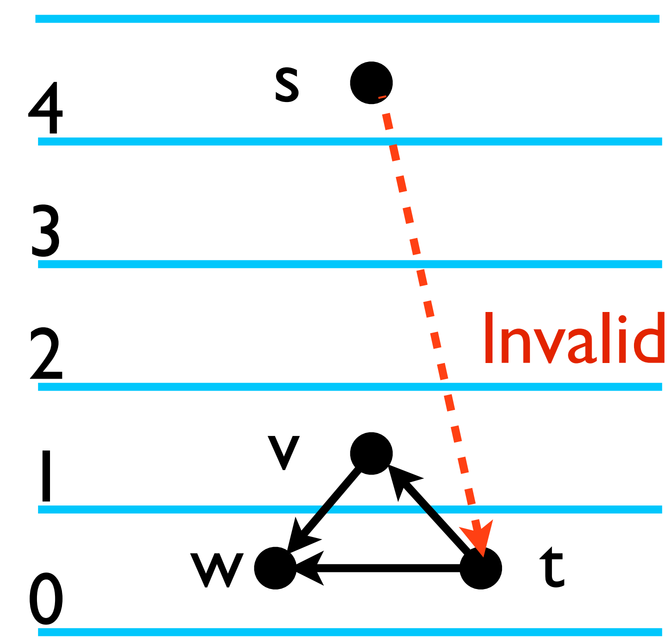
Correctness: Compatible Pre-Flows

Preflow: A function f: E $\rightarrow$ R is a preflow if:

1. **Capacity Constraints:** 0 <= f(e) <= c(e)
2. Instead of conservation constraints:

$$\sum_{e\ into\ v} f(e) - \sum_{e\ out\ of\ v} f(e) \geq 0$$

Excess(v) = $\sum_{e\ into\ v} f(e) - \sum_{e\ out\ of\ v} f(e)$



Preflow f and labeling h are compatible if:

1. h(s) = n, h(t) = 0
2. For all edges (v, w) in the residual graph G_f, h(v) <= h(w) + 1

PreFlow Push: Correctness

Start with labeling: $h(s) = n, h(t) = 0, h(v) = 0$, for other v
Start with preflow f: $f(e) = c(e)$ for $e = (s, v)$, $f(e) = 0$, ow

While there is a node (other than t) with positive excess
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Preflow f and labeling l are **compatible** if:
1. $h(s) = n$, $h(t) = 0$
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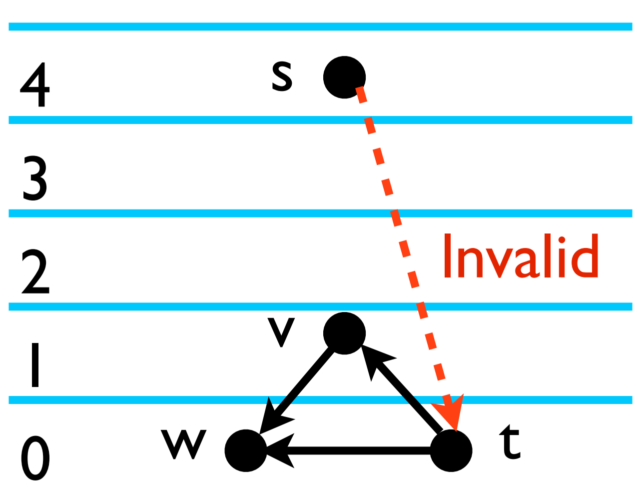
Invariant: Preflow f and labeling h are always compatible over the Preflow-Push algorithm



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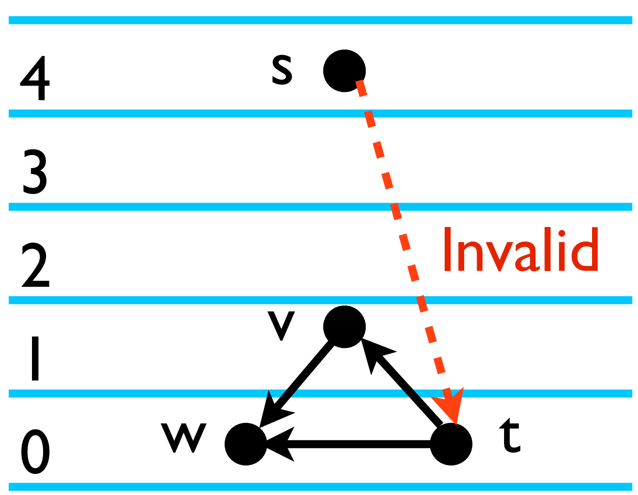
Proof: By induction.



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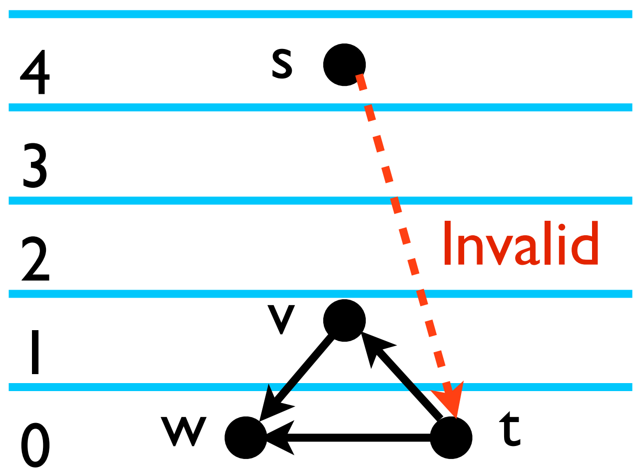
Invariant: Preflow f and labeling h are always compatible over the Preflow-Push algorithm

Proof: By induction. Initially, compatible, as G_f has no (s, v) edges



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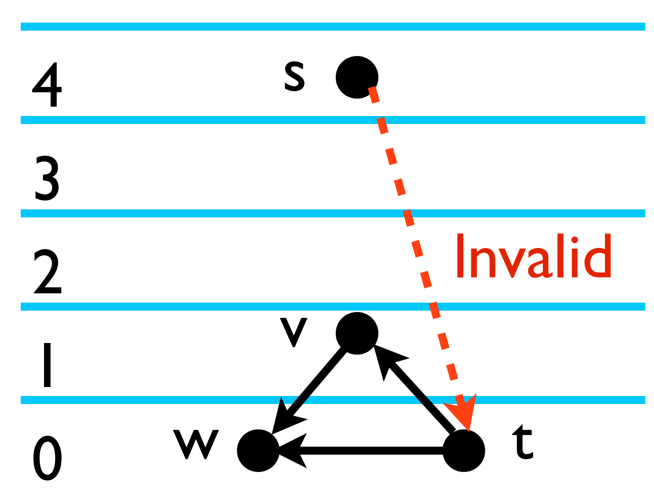

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Proof: By induction. Initially, compatible, as G_f has no (s, v) edges
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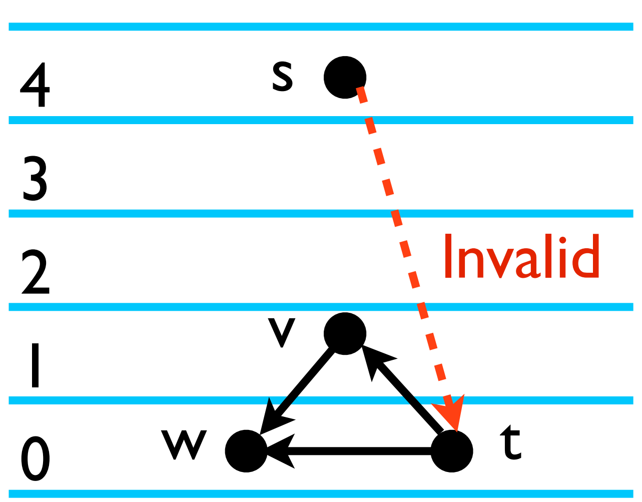
  - **Relabel:** Labels increase only if no downward edges in G_f



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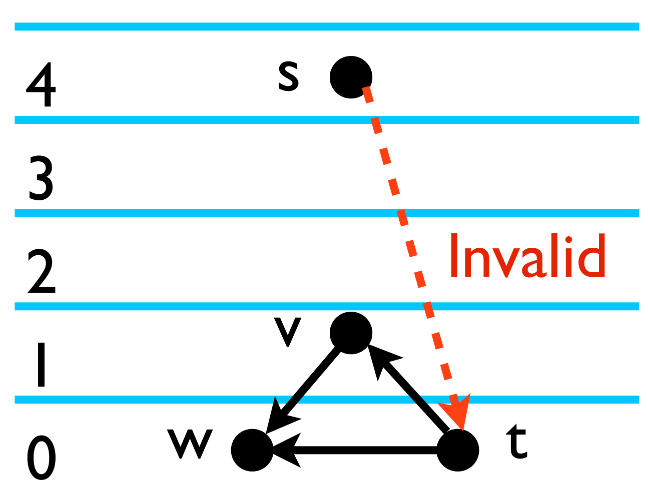

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Suppose f and h are compatible at time t. At time t+1:
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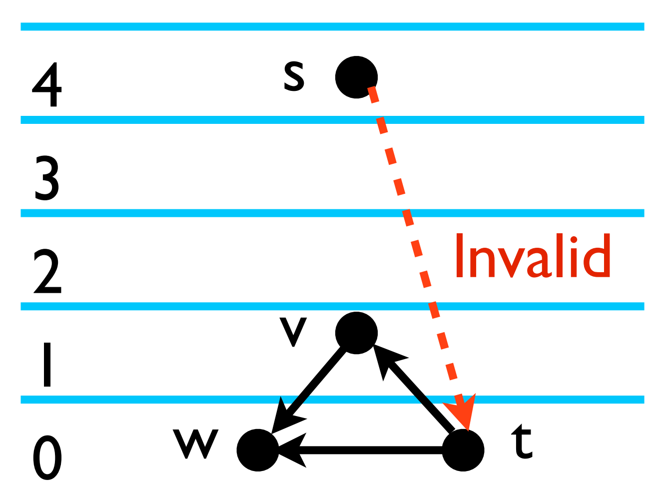

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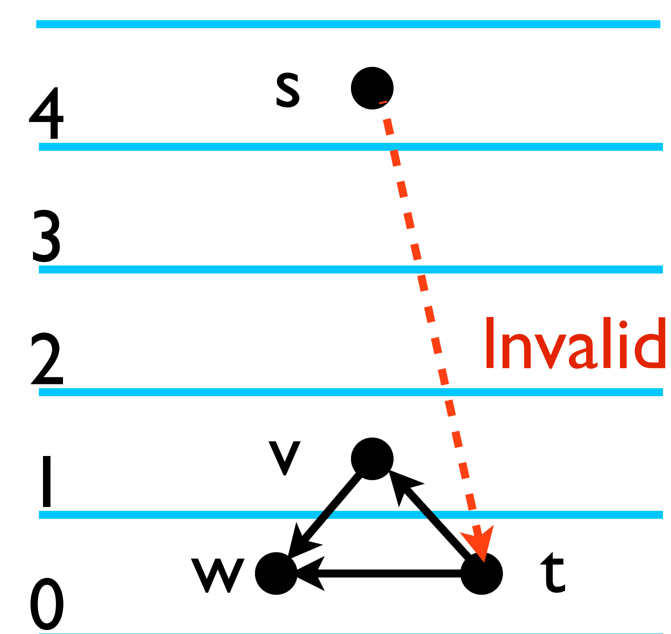
Properties of Compatible PreFlows

Preflow f and labeling h are compatible if:

1. h(s) = n, h(t) = 0
2. For all edges (v, w) in the residual graph G_f, h(v) <= h(w) + 1

Property 1: If preflow f and labeling h are compatible, then there is no s-t path in G_f



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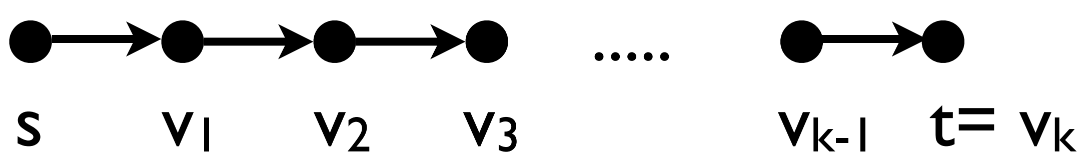
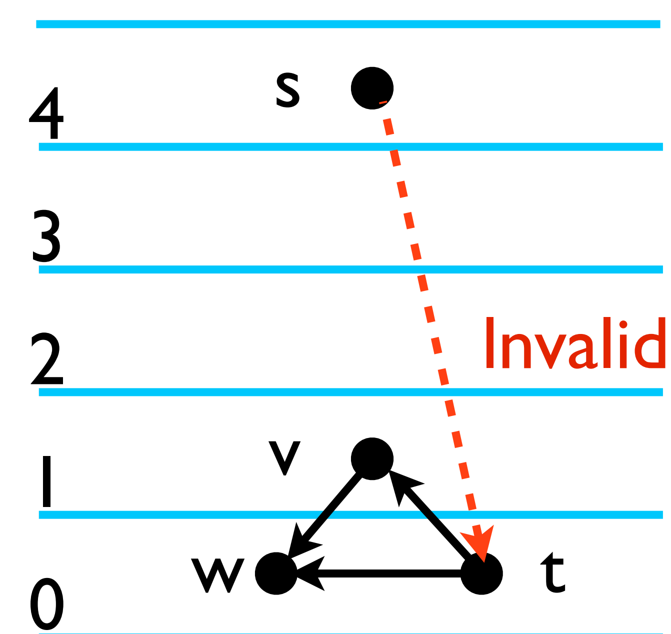
Proof: Suppose there is an s-t path in G_f





Properties of Compatible PreFlows

Preflow f and labeling h are compatible if:
1. $h(s) = n, h(t) = 0$
2. For all edges (v, w) in the residual graph G_f, $h(v) <= h(w) + 1$

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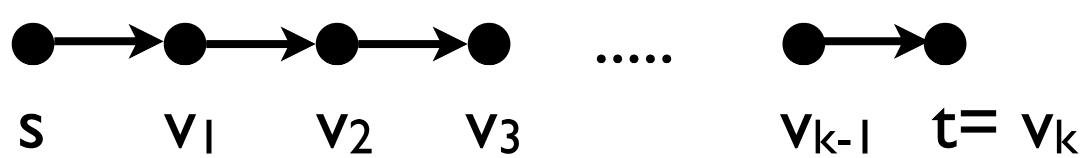
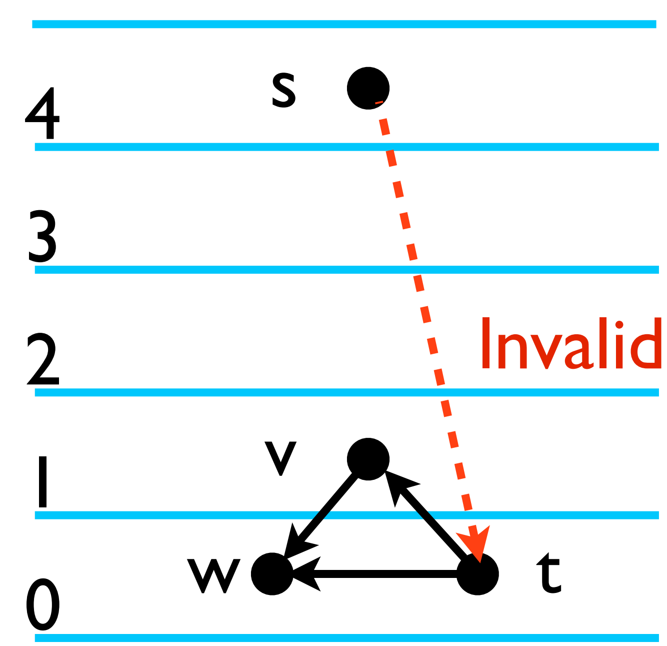


Due to compatibility,
$h(v_1) >= h(s) - 1 = n - 1$



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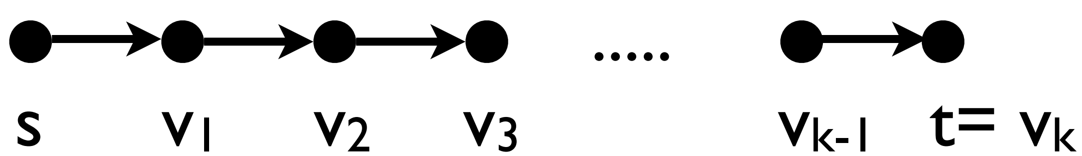
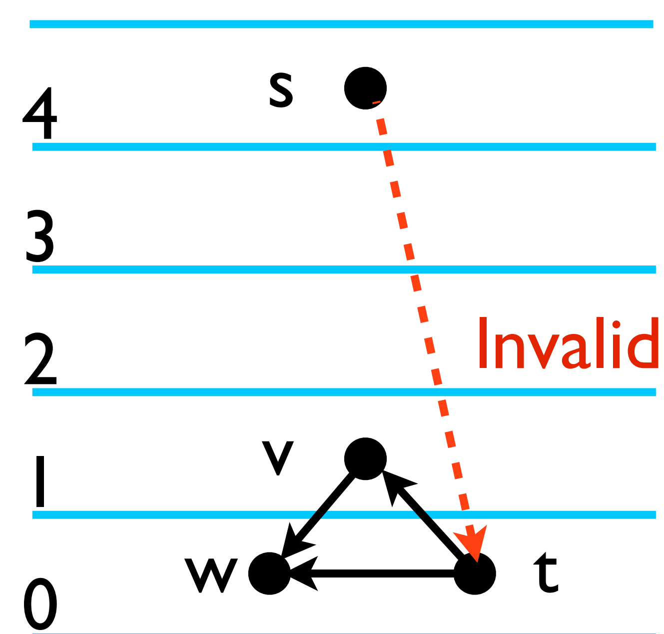


Due to compatibility,

$h(v_1) >= h(s) - 1 = n - 1$

$h(v_2) >= h(v_1) - 1 >= n - 2$

...



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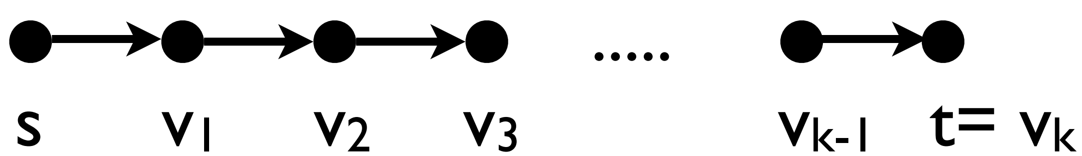

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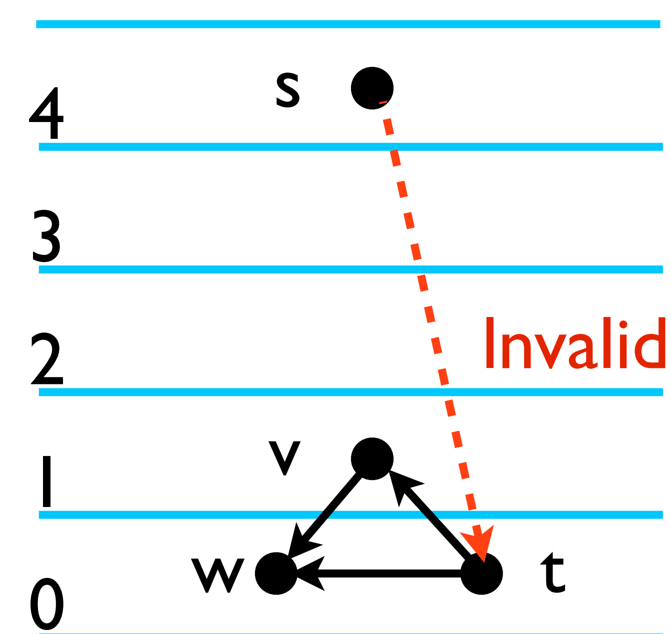
$h(t) = h(v_k) - 1 >= n - k > 0$ (as $k < n$)

Contradiction!



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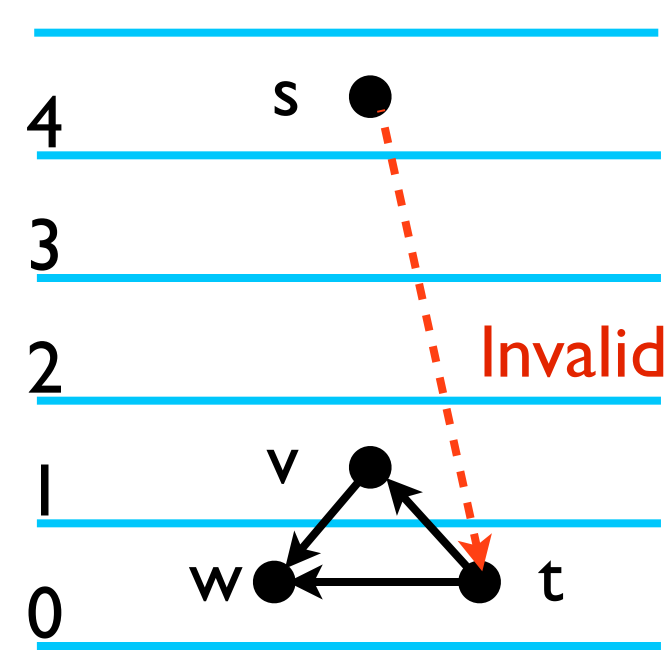
Property 2: If **flow** f and labeling h are compatible, then f is a max flow

Proof: From Property 1 and properties of max flow



Correctness: Proof Outline

Three Steps:

- Compatibility: Show that the preflow f and the labeling h maintained by the algorithm always obeys a **compatibility** property

- If a flow f is compatible with some labeling, then f is a max-flow

- Preflow-push outputs a flow on termination

PreFlow Push: Correctness

Start with labeling: $h(s) = n, h(t) = 0, h(v) = 0$, for other v
Start with preflow f: $f(e) = c(e)$ for $e = (s, v)$, $f(e) = 0$, ow

While there is a node (other than t) with positive excess
    Pick a node v with excess(v) > 0
    If there is an edge (v, w) in E_f s. t. push(v, w) applies
        Push(v, w)
    Else
        Relabel(v)

Push(v, w):
Applies if excess(v) > 0, $h(w) < h(v)$
    $q = \min(\text{excess}(v), c_f(v,w))$
    Add q to f(v, w)

Relabel(v):
Applies if excess(v) > 0 and for all w s.t (v, w) in E_f, $h(w) >= h(v)$
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Invariant: Preflow f and labeling h are always compatible over the Preflow-Push algorithm

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  Then f is a flow!

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Invariant: Preflow f and labeling h are always compatible over the Preflow-Push algorithm

Fact: When Preflow-push stops, f is a **flow**

From **Property 2** of compatible flows, and **Invariant**, f is a max flow

Thus, Preflow-Push correctly outputs a maxflow

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Pre-Flow Push

- Algorithm
- Correctness
- Running Time Analysis

Running Time Analysis: Outline

1. How many Push Ops? Relabel Ops?

2. How to implement Push and Relabel Ops efficiently?

Running Time Analysis: Outline

1. How many Relabel Ops?

   Main Idea: Bound the maximum value of h(v) for any node v, and bound #relabel ops through this