CSE 202: Design and Analysis of Algorithms

Lecture 18

Instructor: Kamalika Chaudhuri
Announcements

• No TA Office Hours on Thu May 26

• Extra Instructor Office Hours Thu May 26 5-6pm (instead of 2:30-3:30pm), CSE 4110

• No class on Monday May 30 (Memorial Day)

• Extra Instructor Office Hours Tue May 31 5-6pm at CSE 4110

• TA Office Hours on Tue May 31 moved to Wed Jun 1, 11-12, B250A

• Pick up graded Homework 3 after class
Randomized Algorithms

- Contention Resolution
- Some Facts about Random Variables
- Global Minimum Cut Algorithm
- Randomized Selection and Sorting
- Max 3-SAT
- Three Concentration Inequalities
- Hashing and Balls and Bins
Hashing and Balls-n-Bins

**Problem:** Given a large set $S$ of elements $x_1, \ldots, x_n$, store them using $O(n)$ space s.t it is easy to determine whether a query item $q$ is in $S$ or not

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<tr>
<th>Table</th>
<th>Linked list of all $x_i$ s.t $h(x_i) = 2$</th>
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**Popular Data Structure:** A Hash table
Hashing and Balls-n-Bins

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![Table with links](image)

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Linked list of all $x_i$ s.t $h(x_i) = 2$

**Popular Data Structure:** A Hash table

**Algorithm:**

1. Pick a completely random function $h : U \rightarrow \{1, \ldots, n\}$
2. Create a table of size $n$, initialize it to null
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What is the query time of the algorithm?
**Hashing and Balls-n-Bins**

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**Average Query Time:** Suppose $q$ is picked at random s.t it is equally likely to hash to 1, .., n. What is the expected query time?
Hashing and Balls-n-Bins

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**Average Query Time:** Suppose $q$ is picked at random s.t it is equally likely to hash to $1, \ldots, n$. What is the expected query time?

$$\text{Expected Query Time} = \sum_{i=1}^{n} \Pr[q \text{ hashes to location } i] \cdot (\text{length of list at } T[i])$$
Hashing and Balls-n-Bins

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**Expected Query Time**

\[
E[q] = \sum_{i=1}^{n} Pr[q \text{ hashes to location } i] \cdot (\text{length of list at } T[i])
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\[
= \frac{1}{n} \sum_{i} \text{(length of list at } T[i])
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Hashing and Balls-n-Bins

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\text{Expected Query Time} = \sum_{i=1}^{n} \Pr[q \text{ hashes to location } i] \cdot (\text{length of list at } T[i])
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$$
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$$
**Problem:** Given a large set \( S \) of elements \( x_1, \ldots, x_n \), store them using \( O(n) \) space s.t it is easy to determine whether a query item \( q \) is in \( S \) or not

**Algorithm:**
1. Pick a completely random function \( h \)
2. Create a table of size \( n \), initialize it to null
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**Worst Case Query Time:** For any \( q \), what is the query time? (with high probability over the choice of hash functions)
Hashing and Balls-n-Bins

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Worst Case Query Time: For any q, what is the query time? (with high probability over the choice of hash functions)

Equivalent to the following Balls and bins Problem:
Suppose we toss n balls u.a.r into n bins. What is the max #balls in a bin with high probability?

With high probability (w.h.p) = With probability $1 - 1/poly(n)$
Balls and Bins, again

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

Some Facts:
Balls and Bins, again

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

Some Facts:

1. The expected load of each bin is 1
2. What is the probability that each bin has load 1?
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\text{Probability} = \frac{\# \text{ permutations}}{\# \text{ ways of tossing } n \text{ balls to } n \text{ bins}} = \frac{n!}{n^n}
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Suppose we toss \( n \) balls u.a.r into \( n \) bins. What is the max load of a bin with high probability?

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3. What is the expected \#empty bins?

\[
\text{Pr}[\text{Bin } i \text{ is empty}] = \left(1 - \frac{1}{n}\right)^n
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E[\# \text{ empty bins}] = n \left(1 - \frac{1}{n}\right)^n = \Theta(n)
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Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

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   $$E[\# \text{ empty bins}] = n \left(1 - \frac{1}{n}\right)^n = \Theta(n) \quad (\text{for } n \geq 2)$$
Balls and Bins

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

Let $X_i = \#\text{balls in bin } i$
Balls and Bins

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

Let $X_i = \#\text{balls in bin } i$

$$\Pr(X_i \geq t) \leq \binom{n}{t} \frac{1}{n^t}$$
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From Fact

Fact: If $n \geq k$

$$\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{ne}{k} \right)^k$$
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Let $X_i =$ #balls in bin $i$

$$\Pr(X_i \geq t) \leq \left(\frac{n}{t}\right)^{1-n^t} \leq \left(\frac{ne}{t}\right)^{t} \frac{1}{n^t} \leq \left(\frac{e}{t}\right)^t$$

From **Fact**

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Would like this for whp condition

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Let $t = \frac{c \log n}{\log \log n}$ for constant $c$
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$$\log \left(\frac{t}{e}\right)^t = t \log t - t$$
Balls and Bins

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

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\Pr(X_i \geq t) \leq \left( \frac{n}{t} \right)^{\frac{1}{n^t}} \leq \left( \frac{ne}{t} \right)^{t} \frac{1}{n^t} \leq \left( \frac{e}{t} \right)^{t} \leq \frac{1}{n^2}
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From Fact Would like this for whp condition

Let $t = \frac{c \log n}{\log \log n}$ for constant $c$

\[
\log \left( \frac{t}{e} \right)^{t} = t \log t - t = \frac{c \log n}{\log \log n} \cdot (\log c + \log \log n - \log \log \log n)
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For large $n$, this is

$$\geq \frac{1}{2} \log \log n$$
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\]

\[
\geq \frac{c}{2} \log n \geq 2 \log n, \text{ for } c \geq 4
\]

For large \( n \), this is

\[
\geq \frac{1}{2} \log \log n
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Balls and Bins

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

Let $X_i = \#\text{balls in bin } i$

$$\Pr(X_i \geq t) \leq \left(\frac{n}{t}\right)^{1/n} \leq \left(\frac{ne}{t}\right)^{t} \leq t \leq \frac{1}{n^2}$$

From Fact: $$(\frac{n}{k})^k \leq \left(\frac{n}{k}\right) \leq \left(\frac{ne}{k}\right)^k$$

Let $t = \frac{c \log n}{\log \log n}$ for constant $c$

$$\log \left(\frac{t}{e}\right)^t = t \log t - t = \frac{c \log n}{\log \log n} \cdot (c \log c + \log \log n - \log \log \log n) \geq \frac{c}{2} \log n \geq 2 \log n, \text{ for } c \geq 4$$

For large $n$, this is

$$\geq \frac{1}{2} \log \log n$$

Therefore, w.p. $1/n^2$, there are at least $t$ balls in Bin $i$. What is $\Pr(\text{All bins have } \leq t \text{ balls})$?
Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

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From Fact: If $n \geq k$

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Let $t = \frac{c \log n}{\log \log n}$ for constant $c$

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Therefore, w.p. $1/n^2$, there are at least $t$ balls in Bin $i$. What is $\Pr(\text{All bins have } \leq t \text{ balls})$?

Applying Union Bound, $\Pr(\text{All bins have } \leq t \text{ balls}) \geq 1 - 1/n$
Suppose we toss \( n \) balls u.a.r into \( n \) bins. What is the max load of a bin with high probability?

**Fact:** W.p. \( 1 - 1/n \), the maximum load of each bin is at most \( O(\log n / \log \log n) \)

**Fact:** The max loaded bin has \( \left( \log n / 3 \log \log n \right) \) balls with probability at least \( 1 - \text{const.} / n^{(1/3)} \)
Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

**Fact:** W.p. $1 - 1/n$, the maximum load of each bin is at most $O(\log n / \log \log n)$

**Fact:** The max loaded bin has $(\log n / 3 \log \log n)$ balls with probability at least $1 - \text{const.}/n^{(1/3)}$

Let $X_i = \#\text{balls in bin } i$
Balls and Bins

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Let \( X_i = \# \text{balls in bin } i \)

\[
\Pr(X_i \geq t) \geq \binom{n}{t} \frac{1}{n^t} \left(1 - \frac{1}{n}\right)^{n-t}
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Balls and Bins

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Let \( X_i = \# \text{balls in bin i} \)

\[
\Pr(X_i \geq t) \geq \binom{n}{t} \left( \frac{1}{n} \right)^t \left( 1 - \frac{1}{n} \right)^{n-t} \geq \left( \frac{n}{t} \right)^t \cdot \frac{1}{n^t} \cdot e^{-1}
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\Pr(X_i \geq t) \geq \binom{n}{t} \frac{1}{n^t} \left(1 - \frac{1}{n}\right)^{n-t} \geq \left(\frac{n}{t}\right)^t \cdot \frac{1}{n^t} \cdot e^{-1} \geq \frac{1}{et^t}
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Let $X_i = \#\text{balls in bin } i$

$$\Pr(X_i \geq t) \geq \binom{n}{t} \frac{1}{n^t} \left(1 - \frac{1}{n}\right)^{n-t} \geq \left(\frac{n}{t}\right)^t \cdot \frac{1}{n^t} \cdot e^{-1} \geq \frac{1}{et^t}$$

At least $1/en^{1/3}$ for $t = \log n/3 \log \log n$
**Balls and Bins**

Suppose we toss \( n \) balls u.a.r into \( n \) bins. What is the max load of a bin with high probability?

**Fact:** W.p. \( 1-1/n \), the maximum load of each bin is at most \( O(\log n/\log \log n) \)

**Fact:** The max loaded bin has \( (\log n/3\log \log n) \) balls with probability at least \( 1 - \text{const.}/n^{(1/3)} \)

Let \( X_i = \# \text{balls in bin } i \)

\[
\Pr(X_i \geq t) \geq \binom{n}{t} \frac{1}{n^t} \left(1 - \frac{1}{n}\right)^{n-t} \geq \left(\frac{n}{t}\right)^t \cdot \frac{1}{n^t} \cdot e^{-1} \geq \frac{1}{e^{t^2}}
\]

At least \( 1/en^{1/3} \) for \( t = \log n/3 \log \log n \)

Let \( Y_i = 1 \) if bin \( i \) has load \( t \) or more,
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\[ Y = Y_1 + Y_2 + .. + Y_n \]
Balls and Bins

Suppose we toss n balls u.a.r into n bins. What is the max load of a bin with high probability?

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At least $1/en^{1/3}$ for $t = \log n/3 \log \log n$

Let $Y_i = 1$ if bin i has load t or more, = 0 otherwise

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$$\Pr(Y_i = 1) \geq 1/en^{1/3}$$
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Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

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$$Y = Y_1 + Y_2 + \ldots + Y_n$$

$$\Pr(Y_i = 1) \geq 1/en^{1/3}$$

$$\mathbb{E}(Y) \geq n^{2/3}/e$$
Balls and Bins

Suppose we toss $n$ balls u.a.r into $n$ bins. What is the max load of a bin with high probability?

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$$\Pr(X_i \geq t) \geq \left( \frac{n}{t} \right) \left( 1 - \frac{1}{n} \right)^{n-t} \geq \left( \frac{n}{t} \right)^t \cdot \frac{1}{n^t} \cdot e^{-1} \geq \frac{1}{e^t}$$

At least $1/e^{t}$ for $t = \log n / 3 \log \log n$

Let $Y_i = 1$ if bin $i$ has load $t$ or more, $= 0$ otherwise

$Y = Y_1 + Y_2 + \ldots + Y_n$

$$\Pr(Y_i = 1) \geq 1/e^{t}$$

$E(Y) \geq n^{2/3} / e$

$$\Pr(Y = 0) = \Pr(\text{No bin has load } t \text{ or more}) \leq \Pr(|Y - E[Y]| \geq E[Y])$$

Which concentration bound to use?
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Using Chebyshev, $\Pr(|Y - E[Y]| \geq E[Y]) \leq \frac{\text{Var}(Y)}{E(Y)^2}$

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\[
\begin{align*}
= 0 & \text{ otherwise} \\
Y &= Y_1 + Y_2 + \ldots + Y_n
\end{align*}
\]

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Chebyshev
Balls and Bins

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$Y = Y_1 + Y_2 + \ldots + Y_n$

$\Pr(Y_i = 1) \geq 1/\text{en}^{1/3}$

$E(Y) \geq n^{2/3}/e$

$\Pr(Y = 0) = \Pr(\text{No bin has load } \geq t) \leq \Pr(|Y - E[Y]| \geq E[Y]) \leq \text{Var}(Y)/E(Y)^2$ (Chebyshev)

$\text{Var}[Y] = \text{Var}[(Y_1 + \ldots + Y_n)^2]$ =
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Now if $i$ is not $j$, $Y_i$ and $Y_j$ are negatively correlated, which means that $E[Y_iY_j] < E[Y_i]E[Y_j]$
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$Y = Y_1 + Y_2 + .. + Y_n$

Pr($Y = 0$) = Pr(No bin has load $\geq t$) $\leq$ Pr($|Y - E[Y]| \geq E[Y]$) $\leq$ Var($Y$)/$E(Y)^2$

$E(Y) \geq n^{2/3} / e$

Pr($Y_i = 1$) $\geq$ $1/\text{en}^{1/3}$

Var[$Y$] = Var[(Y$_1$ + .. + Y$_n$)$^2$] = $\sum_i$ Var($Y_i$) + 2 $\sum_{i \neq j}$ (E[$Y_i Y_j$] - E[$Y_i$]E[$Y_j$])

Now if $i$ is not $j$, $Y_i$ and $Y_j$ are negatively correlated, which means that $E[Y_i Y_j] < E[Y_i] E[Y_j]$

Thus, $Var(Y) \leq \sum_{i=1}^{n} Var(Y_i) \leq n \cdot 1$
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Pr($Y_i = 1$) $\geq 1 / en^{1/3}$

$E(Y) \geq n^{2/3} / e$

Pr($Y = 0$) $= \Pr(\text{No bin has load } \geq t) \leq \Pr(|Y - E[Y]| \geq E[Y]) \leq \text{Var}(Y) / E(Y)^2$

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Now if $i$ is not $j$, $Y_i$ and $Y_j$ are negatively correlated, which means that $E[Y_iY_j] < E[Y_i]E[Y_j]$

Thus,

$\text{Var}(Y) \leq \sum_{i=1}^{n} \text{Var}(Y_i) \leq n \cdot 1$

Pr($Y = 0$) $\leq \frac{\text{Var}(Y)}{E(Y)^2} \leq \frac{ne^2}{n^{4/3}} \leq \frac{e^2}{n^{1/3}}$
Randomized Algorithms

- Contention Resolution
- Some Facts about Random Variables
- Global Minimum Cut Algorithm
- Randomized Selection and Sorting
- Max 3-SAT
- Three Concentration Inequalities
- Hashing and Balls and Bins
  - The Power of Two Choices
The Power of Two Choices

**Problem:** Given a large set $S$ of elements $x_1, \ldots, x_n$, store them using $O(n)$ space s.t it is easy to determine whether a query item $q$ is in $S$ or not

![Diagram of a table with linked lists](image)

**Algorithm:**
1. Pick **two** completely random functions $h_1 : \mathcal{U} \rightarrow \{1, \ldots, n\}$, and $h_2 : \mathcal{U} \rightarrow \{1, \ldots, n\}$
2. Create a table of size $n$, initialize it to null
3. Store $x_i$ at linked list at position $h_1(x_i)$ or $h_2(x_i)$, whichever is shorter
4. For a query $q$, look at the linked list at location $h_1(q)$ and $h_2(q)$ of table to see if $q$ is there
The Power of Two Choices

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<table>
<thead>
<tr>
<th>Table</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
</table>

Linked list of all \( x_i \) s.t \( h(x_i) = 2 \)

**Algorithm:**
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4. For a query \( q \), look at the linked list at location \( h_1(q) \) and \( h_2(q) \) of table to see if \( q \) is there

**Equivalent to the following Balls and Bins Problem:** Toss \( n \) balls into \( n \) bins. For each ball, pick two bins u.a.r and put the ball into the lighter of the two bins.

What is the worst case query time?
Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

We will prove this for the rest of class
Power of Two Choices

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Main Idea (Not a formal Proof):

If $a_i$ = fraction of bins with $i$ or more balls, then, fraction of bins with $i+1$ or more balls is:
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If $a_i$ = fraction of bins with i or more balls, then, fraction of bins with $i+1$ or more balls is:

$$a_{i+1} \approx a_i^2$$

(To put a ball in a bin with i or more balls, need to pick two such bins out of n)
Power of Two Choices

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If $a_i = \text{fraction of bins with } i \text{ or more balls}$, then, fraction of bins with $i+1$ or more balls is:

$a_{i+1} \approx a_i^2$  \hspace{1cm} (To put a ball in a bin with $i$ or more balls, need to pick two such bins out of $n$)

$a_2 \leq \frac{1}{2}$  \hspace{1cm} $a_i \approx \frac{1}{2^{2i-1}} < \frac{1}{n^2}$ for $i = O(\log \log n)$
Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

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If $a_i =$ fraction of bins with $i$ or more balls, then, fraction of bins with $i+1$ or more balls is:

$a_{i+1} \approx a_i^2$  (To put a ball in a bin with $i$ or more balls, need to pick two such bins out of n)

$a_2 \leq \frac{1}{2}$  $a_i \approx \frac{1}{2^{2i-1}} < \frac{1}{n^2}$ for $i = O(\log \log n)$

Now we prove this formally. But what makes the proof hard?
Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

**Fact:** Let $X_1..X_n$ be independent random variables (rv), $Y_1..Y_n$ be rv s.t $Y_i$ depends on $X_1..X_{i-1}$

If $\Pr(Y_i = 1|X_1, \ldots, X_{i-1}) \leq p$, then $\Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(Bin(n, p) \geq k)$
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Define: \( N_i = \# \)bins with load \( i \) or more
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$$\Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(\text{Bin}(n, p) \geq k)$$

**Define:** $N_i = \#\text{bins with load } i \text{ or more}$

**Lemma 1:** For any $a$,

$$\Pr(N_{i+1} > t \mid N_i \leq an) \leq \frac{\Pr(\text{Bin}(n, a^2) > t)}{\Pr(N_i \leq an)}$$
Power of Two Choices

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Lemma 1: For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

Proof: For any $j$, let $X_j = k$ if ball picks in bin $k$ in a toss. Are $X_j$s independent?
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Lemma 1: For any a, $\Pr(N_i+1 > t|N_i \leq an) \leq \frac{\Pr(\text{Bin}(n, a^2) > t)}{\Pr(N_i \leq an)}$

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Define: $Y_j = 1$ if (ball j lands in bin with current load $\geq i+1$) and ($N_i$ at time j-1 $\leq an$) = 0 otherwise
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Define: \( Y_j = 1 \) if \( (\text{ball j lands in bin with current load } \geq i+1) \) and \( (N_i \text{ at time j-1 } \leq an) \) event A

= 0 otherwise event B
Power of Two Choices

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Y_j depends on $X_1..X_{j-1}$.
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$Y_j$ depends on $X_1..X_{j-1}$. $\Pr(Y_j = 1|X_1,..X_{j-1}) = \Pr(A, B)$
Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** The maximum load of a bin is $O(\log \log n)$, w.p. $1 - 1/n$.

**Fact:** Let $X_1, X_2, \ldots, X_n$ be independent random variables (rv), $Y_1, Y_2, \ldots, Y_n$ be rv s.t. $Y_i$ depends on $X_1, \ldots, X_{i-1}$.

If $\Pr(Y_i = 1|X_1, \ldots, X_{i-1}) \leq p$, then $\Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(\text{Bin}(n, p) \geq k)$.

**Define:** $N_i$ = #bins with load $i$ or more.

**Lemma 1:** For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(\text{Bin}(n, a^2) > t)}{\Pr(N_i \leq an)}$.

**Proof:** For any $j$, let $X_j = k$ if ball picks in bin $k$ in a toss. Are $X_j$s independent?

Define: $Y_j = 1$ if (ball $j$ lands in bin with current load $\geq i+1$) and ($N_i$ at time $j-1 \leq an$) and ($N_i$ at time $j-1 \leq an$) and \text{event A} = 0 otherwise \text{event B}

$Y_j$ depends on $X_1, \ldots, X_{j-1}$.

$\Pr(Y_j = 1|X_1, \ldots, X_{j-1}) = \Pr(\text{A, B}) = \Pr(\text{A|B}) \Pr(B)$.
Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - \frac{1}{n}$

**Fact:** Let $X_1..X_n$ be independent random variables (rv), $Y_1..Y_n$ be rv s.t $Y_i$ depends on $X_1..X_{i-1}$

If $\Pr(Y_i = 1 | X_1, \ldots, X_{i-1}) \leq p$, then $\Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(Bin(n, p) \geq k)$

**Define:** $N_i = \#bins with load $i$ or more

**Lemma 1:** For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

**Proof:** For any $j$, let $X_j = k$ if ball picks in bin $k$ in a toss. Are $X_j$s independent?

Define: $Y_j = 1$ if (ball $j$ lands in bin with current load $\geq i+1$) and ($N_i$ at time $j-1$ $\leq an$) and $= 0$ otherwise

$Y_j$ depends on $X_1..X_{j-1}$. $\Pr(Y_j = 1 | X_1,..X_{j-1}) = \Pr(A, B) = \Pr(A|B) \Pr(B) \leq \Pr(A|B) \leq a^2$
Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin $= O(\log \log n)$, w.p. $1 - 1/n$

**Fact:** Let $X_1..X_n$ be independent random variables (rv), $Y_1..Y_n$ be rv s.t $Y_i$ depends on $X_1..X_{i-1}$.

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**Define:** $N_i = \#\text{bins with load } i \text{ or more}$

**Lemma 1:** For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

**Proof:** For any $j$, let $X_j = k$ if ball picks in bin $k$ in a toss. Are $X_j$s independent?

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Y$_j$ depends on $X_1..X_{j-1}$. $\Pr(Y_j = 1 | X_1,..X_{j-1}) = \Pr(A, B) = \Pr(A|B) \Pr(B) \leq \Pr(A|B) \leq a^2$

$\Pr(N_{i+1} > t | N_i \leq an) = \frac{\Pr(N_{i+1} > t, N_i \leq an)}{\Pr(N_i \leq an)}$
Power of Two Choices

Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

**Fact:** Let $X_1..X_n$ be independent random variables (rv), $Y_1..Y_n$ be rv s.t $Y_i$ depends on $X_1..X_{i-1}$
If $\Pr(Y_i = 1|X_1, \ldots, X_{i-1}) \leq p$, then $\Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(Bin(n, p) \geq k)$

**Define:** $N_i = \#bins with load $i$ or more

**Lemma 1:** For any $a$, $\Pr(N_{i+1} > t|N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

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$Y_j$ depends on $X_1..X_{j-1}$. $\Pr(Y_j = 1|X_1,..X_{j-1}) = \Pr(A, B) = \Pr(A|B) \Pr(B) \leq \Pr(A|B) \leq a^2$

$\Pr(N_{i+1} > t|N_i \leq an) = \frac{\Pr(N_{i+1} > t, N_i \leq an)}{\Pr(N_i \leq an)} = \frac{\Pr(\sum_{j=1}^{n} Y_j > t)}{\Pr(N_i \leq an)}$
Power of Two Choices

Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

\[ \text{Fact: Max load of a bin} = O(\log \log n), \text{ w.p. } 1 - 1/n \]

\[ \text{Fact: Let } X_1..X_n \text{ be independent random variables (rv), } Y_1..Y_n \text{ be rv s.t } Y_i \text{ depends on } X_1..X_{i-1} \]

If \( \Pr(Y_i = 1 | X_1, \ldots, X_{i-1}) \leq p \), then \( \Pr(\sum_{i=1}^{n} Y_i \geq k) \leq \Pr(\text{Bin}(n, p) \geq k) \)

Define: \( N_i = \# \text{bins with load } i \text{ or more} \)

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(\text{Bin}(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Proof:** For any \( j \), let \( X_j = k \) if ball picks in bin \( k \) in a toss. Are \( X_j \)s independent?

Define: \( Y_j = 1 \) if (ball \( j \) lands in bin with current load \( \geq i+1 \)) and \( N_i \) at time \( j-1 \leq an \) \( = 0 \) otherwise \( \quad \) event A \( \quad \) event B

\( Y_j \) depends on \( X_1..X_{j-1} \). \( \Pr(Y_j = 1 | X_1,..X_{j-1}) = \Pr(A, B) = \Pr(A|B) \Pr(B) \leq \Pr(A|B) \leq a^2 \)

\( \Pr(N_{i+1} > t | N_i \leq an) = \frac{\Pr(N_{i+1} > t, N_i \leq an)}{\Pr(N_i \leq an)} = \frac{\Pr(\sum_{j=1}^{n} Y_j > t)}{\Pr(N_i \leq an)} \leq \frac{\Pr(\text{Bin}(n, a^2) > t)}{\Pr(N_i \leq an)} \) from Fact
Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

**Define:** $N_i = \#\text{bins with load } i \text{ or more}$

**Lemma 1:** For any $a$, \[ \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \]

**Lemma 2:** If $a^2 \geq 6 \ln n/n$ then \[ \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \]
If $a^2 < 6 \ln n/n$ then \[ \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \]
Power of Two Choices

Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = \( O(\log \log n) \), w.p. 1-1/n

**Define:** \( N_i = \# \text{bins with load } i \text{ or more} \)

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t \mid N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Lemma 2:** If \( a^2 \geq 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)
If \( a^2 < 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

**Proof:** Use Chernoff Bounds. What is \( X \) and what is \( X_i \)?

**Chernoff Bounds:**
Let \( X_1, \ldots, X_n \) be independent 0/1 rvs, \( X = X_1 + \ldots + X_n \). For \( t > 0 \),
\[
\Pr(X \geq (1 + t)E[X]) \leq \left( \frac{e^t}{(1 + t)^{(1+t)}} \right)^{E[X]}
\]
For \( t < 2e - 1 \),
\[
\Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2E[X]/4)
\]
Power of Two Choices

Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = \( O(\log \log n) \), w.p. \( 1 - 1/n \)

**Define:** \( N_i = \# \) bins with load \( i \) or more

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Lemma 2:** If \( a^2 \geq 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)
If \( a^2 < 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

**Proof:** Use Chernoff Bounds. What is \( X \) and what is \( X_i \) ?

For the first part, \( E[X] = na^2 \)

**Chernoff Bounds:**
Let \( X_1, \ldots, X_n \) be independent 0/1 rvs, \( X = X_1 + \ldots + X_n \). For \( t > 0 \),

\[
\Pr(X \geq (1 + t)E[X]) \leq \left( \frac{e^t}{(1 + t)^{(1+t)}} \right)^{E[X]}
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For \( t < 2e - 1 \),

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Power of Two Choices

Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

Fact: Max load of a bin = \( O(\log \log n) \), w.p. \( 1 - 1/n \)

Define: \( N_i = \#\) bins with load \( i \) or more

Lemma 1: For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

Lemma 2: If \( a^2 \geq 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)

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Proof: Use Chernoff Bounds. What is \( X \) and what is \( X_i \)?

For the first part, \( E[X] = na^2 \) \( t = 1 \)

Chernoff Bounds:
Let \( X_1, \ldots, X_n \) be independent 0/1 rvs, \( X = X_1 + \ldots + X_n \). For \( t > 0 \),
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Power of Two Choices

Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

Fact: Max load of a bin = O(log log n), w.p. 1-1/n

Define: \( N_i = \# \text{bins with load } i \text{ or more} \)

Lemma 1: For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

Lemma 2: If \( a^2 \geq 6 \ln n / n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)
If \( a^2 < 6 \ln n / n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

Proof: Use Chernoff Bounds. What is \( X \) and what is \( X_i \)?

For the first part, \( E[X] = na^2 \ t = 1 \)
\( \Pr(Bin(n, a^2) > 2na^2) \leq (e/4)^{na^2} \)

Chernoff Bounds:
Let \( X_1, \ldots, X_n \) be independent 0/1 rvs, \( X = X_1 + \ldots + X_n \). For \( t > 0 \),
\[ \Pr(X \geq (1 + t)E[X]) \leq \left( \frac{e^t}{(1 + t)^{(1+t)}} \right)^{E[X]} \]
For \( t < 2e - 1 \),
\[ \Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2E[X]/4) \]
Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $\Theta(\log \log n)$, w.p. $1 - 1/n$

**Define:** $N_i = \#$bins with load $i$ or more

**Lemma 1:** For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

**Lemma 2:** If $a^2 \geq 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2$

If $a^2 < 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2$

**Proof:** Use Chernoff Bounds. What is $X$ and what is $X_i$?

For the first part, $E[X] = n a^2$, $t = 1$

$\Pr(Bin(n, a^2) > 2na^2) \leq \left(\frac{e^t}{(1+t)(1+t)}\right)^{E[X]}$

For $t < 2e - 1$,

$\Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2E[X]/4)$
Power of Two Choices

Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

Fact: Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

Define: $N_i = \#\text{bins with load } i \text{ or more}$

Lemma 1: For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

Lemma 2: If $a^2 \geq 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2$
If $a^2 < 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2$

Proof: Use Chernoff Bounds. What is $X$ and what is $X_i$?
For the first part, $E[X] = na^2$, $t = 1$
$\Pr(Bin(n, a^2) > 2na^2) \leq (e/4)^{na^2} \leq (e/4)^{6\ln n} \leq 1/n^2$
For the second part, $E[X] < 6 \ln n$

Chernoff Bounds:
Let $X_1, .., X_n$ be independent 0/1 rvs, $X = X_1 + .. + X_n$. For $t > 0$,
$\Pr(X \geq (1 + t)E[X]) \leq \left(\frac{e^t}{(1 + t)(1 + t)}\right)^{E[X]}$
For $t < 2e - 1$,
$\Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2E[X]/4)$
Power of Two Choices

Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

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**Define:** \( N_i = \# \text{bins with load } i \text{ or more} \)

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Lemma 2:** If \( a^2 \geq 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)

If \( a^2 < 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

**Proof:** Use Chernoff Bounds. What is \( X \) and what is \( X_i \)?

For the first part, \( E[X] = na^2 \) \( t = 1 \)

\( \Pr(Bin(n, a^2) > 2na^2) \leq (e/4)^{na^2} \leq (e/4)^{6\ln n} \leq 1/n^2 \)

For the second part, \( E[X] < 6 \ln n \) so \( t > 10 \ln n/E[X] \)

**Chernoff Bounds:**
Let \( X_1, ..., X_n \) be independent 0/1 rvs, \( X = X_1 + .. + X_n \). For \( t > 0 \),

\( \Pr(X \geq (1 + t)E[X]) \leq \left( \frac{e^t}{(1 + t)^{(1 + t)}} \right)^{E[X]} \)

For \( t < 2e - 1 \),

\( \Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2E[X]/4) \)
Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

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\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}
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\]

For the second part, \( E[X] < 6 \ln n \) so \( t > 10 \ln n / E[X] \)
\[
\Pr(Bin(n, a^2) > 16 \ln n) \leq \exp(-(10 \ln n / E[X])^2 \cdot E[X]/4)
\]

**Chernoff Bounds:**
Let \( X_1, \ldots, X_n \) be independent 0/1 rvs, \( X = X_1 + \ldots + X_n \). For \( t > 0 \),
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\]

For \( t < 2e - 1 \),
\[
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Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = $O(\log \log n)$, w.p. $1 - 1/n$

**Define:** $N_i$ = #bins with load $i$ or more

**Lemma 1:** For any $a$, 
$$\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$$

**Lemma 2:** If
- $a^2 \geq 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2$
- If $a^2 < 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2$

**Proof:** Use Chernoff Bounds. What is $X$ and what is $X_i$?

For the first part, $E[X] = na^2$  $t = 1$
$$\Pr(Bin(n, a^2) > 2na^2) \leq (e/4)^{na^2} \leq (e/4)^{6\ln n} \leq 1/n^2$$

For the second part, $E[X] < 6 \ln n$ so $t > 10 \ln n / E[X]$
$$\Pr(Bin(n, a^2) > 16 \ln n) \leq \exp(-(10 \ln n / E[X])^2 \cdot E[X]/4) \leq \exp(-25 \ln^2 n / E[X])$$

**Chernoff Bounds:**
Let $X_1, ..., X_n$ be independent 0/1 rvs, $X = X_1 + .. + X_n$. For $t>0$,
$$\Pr(X \geq (1+t)E[X]) \leq \left(\frac{e^t}{(1+t)(1+t)}\right)^{E[X]}$$

For $t < 2e - 1$,
$$\Pr(X \geq (1+t)E[X]) \leq \exp(-t^2E[X]/4)$$
**Power of Two Choices**

Toss $n$ balls into $n$ bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

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**Define:** $N_i = \#\text{bins with load } i \text{ or more}$

**Lemma 1:** For any $a$,  
$$\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$$

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**Proof:** Use Chernoff Bounds. What is $X$ and what is $X_i$?

For the first part, $E[X] = na^2$, $t = 1$

$$\Pr(Bin(n, a^2) > 2na^2) \leq (e/4)^{na^2} \leq (e/4)^{6\ln n} \leq 1/n^2$$

For the second part, $E[X] < 6\ln n$ so $t > 10\ln n/E[X]$

$$\Pr(Bin(n, a^2) > 16\ln n) \leq \exp(-(10\ln n/E[X])^2 \cdot E[X]/4) \leq \exp(-25\ln^2 n/E[X]) \leq e^{-4\ln n} \leq n^{-2}$$

**Chernoff Bounds:**
Let $X_1, ..., X_n$ be independent 0/1 rvs, $X = X_1 + .. + X_n$. For $t > 0$,

$$\Pr(X \geq (1 + t)E[X]) \leq \left(\frac{e^t}{(1 + t)^{(1+t)}}\right)^{E[X]}$$

For $t < 2e - 1$,

$$\Pr(X \geq (1 + t)E[X]) \leq \exp(-t^2 E[X]/4)$$
Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = \( O(\log \log n) \), w.p. 1- \( 1/n \)

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**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

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Fact: Max load of a bin = $O(\log \log n)$, w.p. 1 - 1/n

Define: $N_i = \#\text{bins with load } i \text{ or more}$

Lemma 1: For any $a$, $\Pr(N_{i+1} > t|N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

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Define: Sequence of $a_i$ values: $a_4 = 1/4 \quad a_{i+1} = 2a_i^2 \quad E_i = \text{good event } (N_i \leq a_i n)$
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Proof: By Induction. Base Case: \( i = 4 \). \( \Pr(\text{not}(E_4)) = 0 \), as at most \( n/4 \) bins can have \( \geq 4 \) balls
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But \( N_{I^*+1} \) or \( N_{I^*+2} \) may still be large, so we need to bound them
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Let $I^* = \min_i \{a_i^2 \leq 6 \ln n/n\}$ Then, $\Pr(not(E_{I^*})) \leq I^*/n^2 \leq 1/n$

$\Pr(N_{I^*+1} > 16 \ln n) \leq 2/n$

$\Pr(N_{I^*+2} \geq 1) \leq \Pr(Bin(n, \frac{(16 \ln n)^2}{n}) \geq 1) + \Pr(N_{I^*+1} > 16 \ln n)$

$\leq E(Bin(n, \frac{(16 \ln n)^2}{n})) + 2/n$
Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = \( O(\log \log n) \), w.p. \( 1-1/n \)

**Define:** \( N_i = \# \text{bins with load } i \text{ or more} \)

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Lemma 2:** If \( a^2 \geq 6 \ln n / n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)
If \( a^2 < 6 \ln n / n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

**Define:** Sequence of \( a_i \) values: \( a_4 = 1/4 \quad a_{i+1} = 2a_i^2 \) \( E_i = \text{good event } (N_i \leq a_i n) \)

**Lemma 3:** If \( a_i^2 \geq 6 \ln n / n \) then \( \Pr(\text{not}(E_{i+1})) \leq (i + 1)/n^2 \)
Let \( I^* = \min_i \{a_i^2 \leq 6 \ln n / n \} \) Then, \( \Pr(\text{not}(E_{I^*})) \leq I^*/n^2 \leq 1/n \)
\( \Pr(N_{I^*+1} > 16 \ln n) \leq 2/n \)
\( \Pr(N_{I^*+2} \geq 1) \leq \Pr(Bin(n, \left(\frac{16 \ln n}{n}\right)^2) \geq 1) + \Pr(N_{I^*+1} > 16 \ln n) \)
\( \leq E(Bin(n, \left(\frac{16 \ln n}{n}\right)^2)) + 2/n \leq O(\log^2 n / n) \)
Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

Fact: Max load of a bin = $O(\log \log n)$, w.p. 1 - 1/n

Define: $N_i$ = #bins with load i or more

Lemma 1: For any $a$, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

Lemma 2: If $a^2 \geq 6 \ln n/n$ then $\Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2$
If $a^2 < 6 \ln n/n$ then $\Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2$

Define: Sequence of $a_i$ values: $a_4 = 1/4 \quad a_{i+1} = 2a_i^2 \quad E_i$ = good event ($N_i \leq a_i n$)

Lemma 3: If $a_i^2 \geq 6 \ln n/n$ then $\Pr(\text{not}(E_{i+1})) \leq (i + 1)/n^2$
Let $I^* = \min_i \{a_i^2 \leq 6 \ln n/n\}$ Then, $\Pr[\text{not}(E_{I^*})] \leq I^*/n^2 \leq 1/n$

Lemma 4: Wp $(1 - \log^2 n/n)$, there are no bins with load $I^* + 2$ or more
Power of Two Choices

Toss \( n \) balls into \( n \) bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?

**Fact:** Max load of a bin = \( O(\log \log n) \), w.p. 1 - 1/\( n \)

**Define:** \( N_i = \#\text{bins with load } i \text{ or more} \)

**Lemma 1:** For any \( a \), \( \Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)} \)

**Lemma 2:** If \( a^2 \geq 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2 \)
If \( a^2 < 6 \ln n/n \) then \( \Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2 \)

**Define:** Sequence of \( a_i \) values: \( a_4 = 1/4 \quad a_{i+1} = 2a_i^2 \quad E_i = \text{good event } (N_i \leq a_i n) \)

**Lemma 3:** If \( a_i^2 \geq 6 \ln n/n \) then \( \Pr(not(E_{i+1})) \leq (i + 1)/n^2 \)
Let \( I^* = \min_i \{a_i^2 \leq 6 \ln n/n\} \) Then, \( \Pr(not(E_{I^*})) \leq I^*/n^2 \leq 1/n \)

** Lemma 4:** Wp \( (1 - \log^2 n/n) \), no bins with load \( I^* + 2 \) or more
Can be shown that \( a_i \approx \frac{1}{22^i} \) from which \( I^* = O(\log \log n) \)