General Problem

Dynamic programming (DP) is an efficient approach to solving problems where a recursive, backtracking solution ends up repeatedly solving many common subproblems. Recognizing this, DP systematically solves every possible subproblem, but comes out ahead of backtracking by only solving these problems once. Finding a DP algorithm usually involves the following steps:

1. Start with a simple recursive (backtracking) algorithm.
2. Identify repeated subproblems.
3. Name these subproblems, and store their solutions by name in an array.
4. Find a top-down order of dependencies between these subproblems.
5. Invert this order to get a bottom-up order.

The resulting algorithm will typically have the following form:

1. Initialize array.
2. Fill in base cases.
3. In bottom-up order, solve all subproblems.
4. Return the solution to the final subproblem (i.e. the problem).

Example: Bench Spacing

Input: An array of costs \( C[1..n] \) and a maximum separation \( k \).
Output: A set of indices \( i_1 < \ldots < i_m \) such that \( i_{j-1} + k \geq i_j \).
Goal: Minimize the total cost \( \sum_j C[i_j] \).

A backtracking algorithm for this is:
BTBC(C[1..n], k)
if n < k
    return 0
else
    Cmin <- Inf
    for i = 1 .. k
        Cmin <- min(Cmin, C[i] + BTBC(C[i+k .. n], k))
    return Cmin

The running time for this algorithm is

\[ T(n) = T(n-1) + \ldots + T(n-k-1) \leq kT(n-1) \leq k^n \]

which is bad. However, by looking at what the recursive algorithm does, we might guess that we could do much better. Specifically, backtracking solves the following subproblems:

\[ C[1,n] \rightarrow C[2,n] \rightarrow C[3,n] \rightarrow \ldots \rightarrow C[4,n] \]
\[ \rightarrow \ldots \rightarrow C[k+1,n] \rightarrow C[k+2,n] \]
\[ \rightarrow C[k,n] \rightarrow C[k+1,n] \rightarrow \ldots \rightarrow C[2k+1,n] \rightarrow \]

Clearly, many of them are repeated. Therefore we should try applying DP:

Repeated subproblems: C[j,n].
Base cases: if \( n - j + 1 < k \), the cost is 0.
Case order: C[j,n] depends on C[j+1,n] \ldots C[j+k,n]. In other words, the subproblems are needed in order of increasing length.

In these terms, our backtracking algorithm is:

BTBC(C[j..n], k)
if n - j + 1 < k
    return 0
else
    Cmin <- Inf
    for i = j .. n - k
        Cmin <- min(Cmin, C[i] + BTBC(C[i+k .. n], k))
    return Cmin

Inverting this to compute the solutions from shortest to longest, the dynamic programming algorithm is
DPBC(C[1..n], k)

A[1 .. n+1]  // array of costs
for j = n-k .. n+1  // Initialize base cases
  A[j] <- 0
for j = n-k-1 .. 1
  A[j] <- Inf
for i = j .. j + k
return A[1]

This algorithm runs in time $O(kn)$ and space $O(n)$ (but it can be made to require space $O(k)$ – how?). Note that the algorithm does not currently find the optimal placements, but only their total cost. For a hint on how it can be extended to find the path, see the following example.

Example problem

k = 3,
Posn  1 2 3 4 5 6 7 8 9 10 11
C  2 7 3 4 3 2 1 4 9 3 5
A  9 8 7 6 4 4 4 3 3 0 0 0
Next  3 4 6 6 7 7 10 10 10 x x

Here Next[i] is an array holding the indices of the next item in the best placement starting at i. For example, Next[9] = 10 because the best placement for positions 9 through 11 is to place a bench at 10, which has cost 3.