All lattices in this assignment are assumed to have full rank. Let $V_n$ be the volume of the unit sphere in $n$-dimensional space. Minkowski’s convex body theorem implies that any $n$-dimensional lattice $L$ has minimum distance at most $\lambda(L) \leq (2/\sqrt[n]{n}) \cdot \det(L)^{1/n}$. Using $V_n > (2/\sqrt{n})^n$ as a lower bound on the volume of a sphere\(^1\), we get that $\lambda(L) < \sqrt{n} \det(L)^{1/n}$, which is the most commonly used form of Minkowski’s theorem.

In this assignment, you are asked to prove that this bound is tight up to constant multiplicative factors.

**Part 1: Finding holes in a lattice**

Show that if a lattice has determinant at least $\det(L) > d^n V_n$, then there exists a point $x$ at distance $d$ from the lattice, i.e., $\|x - y\| > d$ for all lattice vectors $y \in L$. (Hint: pick a basis for the lattice and use a volume argument similar to the one from the proof of Blichfeldt theorem.)

**Part 2: Filling holes**

Let $h$ be a point in space furthest away from (full rank lattice) $L$, and let $\rho = \min\{\|h - y\| : y \in L\}$ the distance to the closest lattice point. ($\rho$ is called the covering radius of the lattice, and $h$ is called a deep hole.) Show that if $x \in L$ is a lattice point closest to $2h$, then the set $L' = L \cup (L + x/2)$ is a lattice with determinant $\det(L') = \det(L)/2$ and minimum distance $\lambda(L') \geq \min\{\lambda(L), \rho/2\}$.

**Part 3: Dense lattices**

Show that for any $n$ there is an $n$-dimensional (full rank) lattice such that $\lambda(L) \geq (1/(2^{n/2}) \cdot \det(L)^{1/n}$. (Hint: consider the quantity $\sigma = \sup \lambda(L)^n / \det(L)$, where $L$ ranges over all $n$-dimensional lattices, assume for contradiction that $\sigma < 1/(2^n V_n)$, and apply parts 1 and 2 to a lattice $L$ satisfying $\sigma/2 < \lambda(L)/\det(L) \leq \sigma$.) This proves that Minkowski’s bound $\lambda(L) \leq (2/\sqrt{n}) \cdot \det(L)^{1/n}$ is tight up to a factor 4.

**Part 4: The volume of the sphere**

Show that the $V_n > (2/\sqrt{n})^n$ lower bound on the volume of the sphere is also tight. Specifically, show that there is a constant $c_d$ such that $V_n \leq (c_d/\sqrt{n})^n$. For this part, you may want to use the fact that for any even $n = 2k$, $V_n = \pi^{n/2}/(n/2)! = \pi^k/k!$. Conclude that Minkowski’s bound $\lambda(L) \leq \sqrt{n} \det(L)^{1/n}$ is tight up to constant factors.

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\(^1\)This lower bound is easily obtained by observing that the unit sphere contains the hypercube $C = \{x : \|x\| \leq 1/\sqrt{n}\}$