1. Consider a vocabulary consisting of a single binary function \(*\) and the constant 1. The following three sentences over this vocabulary are called the axioms of group theory (we use infix notation for \(*\)):
   - \(\forall x \forall y \forall z ((x * y) * z = x * (y * z))\) (associativity)
   - \(\forall x (x * 1 = x)\) (1 is the identity)
   - \(\forall x \exists y (x * y = 1)\) (existence of inverses).

A group is any structure over vocabulary \(*, 1\), satisfying the above sentences. The three axioms form a complete axiomatization of groups: every sentence satisfied by all groups is provable from the axioms. It is also known that the theory of groups is undecidable (i.e. it is undecidable if a sentence is true for all groups). Now compare this with the situation for the integers. When we proved Gödel’s Incompleteness Theorem we showed that if there was a complete axiomatization for the integers then it would be decidable if a sentence is true for the integers. Why doesn’t our proof work for the theory of groups? Explain what is different.

2. Recall that an FO sentence has the finite model property if either it has no model or it has at least one finite model. Show that it is undecidable if an FO sentence has the finite model property.

3. Use Ehrenfeucht-Fraissé games to show that there is no FO sentence over vocabulary \(\langle <, = \rangle\) that distinguishes\(^1\) the real numbers (with ordering on the reals) from the rational numbers (with the ordering on the rationals). Now suppose the vocabulary is extended to include \(+\). Explain informally why the problem is harder in this case.

4. Let \(E\) be a binary relation (providing edges in a graph) and let \(L_n\) denote an interpretation of \(E\) consisting of a simple path of \(n\) elements. Consider the sets of integers of the form \(N_\varphi = \{n \mid L_n \models \varphi\}\), where \(\varphi\) is an FO sentence over vocabulary \(E\). Describe the sets \(N_\varphi\). (In other words, characterize the sets of integers that equal \(N_\varphi\) for some \(\varphi\).)

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\(^1\)Meaning that it is true for one but not the other.