Graph Theory

Spring, 2006
May 29, 2006
Lecture 17
Graph Theory

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Graph

Definition
- **Graph** $G$ consists of a pair $(V, E)$
  - $V$ is a finite set called the *vertices*
  - $E$ is a set of *edges* that join distinct pairs of $V$
- We say two vertices are *adjacent* if there is an edge between them
- We say a vertex is *incident* to the edges connecting to it

Example
- $V = \{a, b, c, d\}$, $E = \{(a, b), (b, c), (c, d), (d, a), (a, c)\}$
Directed Graph (Digraph)

Definition
- Like a graph, but each edge is *directed*
- Can have one edge in each direction between a pair of vertices

Example

```
a
b

\[\rightarrow\]

\[\rightarrow\]

c

\[\leftarrow\]

\[\leftarrow\]

d

e

\[\rightarrow\]

\[\rightarrow\]

f

g
```
A, B, C, D, and E are applying for jobs v, w, x, y, z as shown in graph G.
Can each person be assigned to a different job?
Definitions

**Path**
- A sequence of distinct vertices $P_1$-$P_2$-$\ldots$-$P_n$ where each pair of consecutive vertices has an edge joining them

**Circuit**
- A path from $P_1$ to $P_n$ where there’s also an edge from $P_n$ to $P_1$.

**Connected**
- A graph is connected if there’s a path between all pairs of vertices
Network Reliability Example

Vertices are power stations
Edges are electrical distribution lines

- Which edges need to be removed in order to disconnect the (remaining) graph?

- Which vertices need to be removed in order to disconnect the (remaining) graph?
Definitions

**Degree of a vertex**
- The count of the number of edges adjacent to the vertex

**Vertex cover of a graph G**
- Subset of vertices G’ such that every edge in G is adjacent to at least one vertex of G’

**Edge Cover of a graph G**
- Subset of edges E’ in G such that every vertex in G is incident to at least one edge of E’.
Street Surveillance Example

Vertices are intersections
Edges are streets

- On what intersections should police officers be placed to patrol each edge under surveillance. (Each edge needs an adjacent police officer).
Scheduling Meetings Example

Vertices are committees

Edges are committees with overlapping membership
  ▪ What is the largest set of committees that can meet at the same time?

*(Maximal) Independent set*
  ▪ (Largest) subset of vertices in a graph with no edges between them.
Relationship between Independent Set and Vertex Cover

Given $G=(V, E)$
- $I$ is Independent set iff $V-I$ is a vertex cover

Proof
- $\Rightarrow$
- $\Leftarrow$

Result
- Finding maximal independent set is same problem as finding minimal vertex cover
Influence Example

Vertices are people

Edge from i to j if i influences j

- What is a minimal set of influencers who can spread an idea to everyone?
Isomorphism

$G_1$ is isomorphic to $G_2$ if:

- there is a one-to-one correspondence between $V_1$ and $V_2$ where two vertices are adjacent in $G_1$ iff the corresponding pair of vertices are adjacent in $G_2$
  - $\exists$ bijection $f: V_1 \rightarrow V_2$ such that $(x, y) \in E_1$ iff $(f(x), f(y)) \in E_2$

Intuitively:
- Can redraw $G_1$ so that it looks like $G_2$
Definitions

**Subgraph** $G'$ of $G$
- Subset of edges and vertices of $G$

**Complete graph, $K_n$ of $n$ vertices**
- Graph on $n$ vertices with all vertices adjacent to each other
Isomorphism

Must match:
- Number of vertices
- Number of edges
- Degrees of vertices
- Subgraphs

Example:
Theorem

In any graph, the sum of the degrees of the vertices is equal to twice the number of edges

Corollary:
- The number of vertices of odd degree is even
Bipartite graph

A graph whose vertices can be partitioned into two sets: $V_1$ and $V_2$ where:

- The only edges are between vertices in $V_1$ and $V_2$

**Theorem:**

- $G$ is bipartite iff every circuit in $G$ has even length
Planar graphs

Graphs that can be drawn in the plane with no crossing edges

Plane graph

- A planar depiction of a planar graph
Some non-planar graphs

- Left graph: a - f - b - e - c - d
- Right graph: a - b - c - d - e
Theorem

A graph is planar iff it does not contain a subgraph that is a $K_5$ or $K_{3,3}$ configuration.

Definition

- A $K_{3,3}$ ($K_5$) configuration is a graph that can be obtained from a $K_{3,3}$ ($K_5$) by adding vertices in the middle of some edges.
Definitions

**Multigraph**
- We allow multiple edges between pairs of vertices (and allow loop edges between a single vertex)

**Trail**
- Like a path, sequence of vertices: $V_1-V_2-\ldots-V_n$, except:
  - Vertices can be repeated
  - Edges cannot be repeated

**Cycle**
- Like a trail, but with an edge $(V_n,V_1)$
Konigsberg bridges

**Euler cycle:** A cycle that
- Uses all vertices
- Contains all edges in the multigraph
First theorem of graph theory

A multigraph contains an Euler cycle iff:
- The graph is connected
- Each vertex has even degree

Corollary: A multigraph has an Euler trail iff:
- it is connected
- Exactly two vertices have odd degree