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Day 14

Linear Homogeneous Recurrence Relations with Constant Coefficient

Instructor: Neil Rhodes
Examples of Limited Master Method

\[ T_n = 3T_{n/2} + n \]

\[ T_n = 4T_{n/4} + n \]

\[ T_n = 4T_{n/5} + n \]
Master Method

**Recurrence:**

\[ T(n) = aT\left(\frac{n}{b}\right) + f(n) \]

**Case 1:** \( f(n) \) “larger” than \( n^{\log_b a} \)

\[ f(n) \]

\[ f\left(\frac{n}{b}\right) \quad \cdots \quad f\left(\frac{n}{b^2}\right) \]

\[ f\left(\frac{n}{b^2}\right) \quad \cdots \quad f\left(\frac{n}{b^3}\right) \]

\[ \vdots \]

\[ a^{\log_b n} = n^{\log_b a} \]

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**Day 3 – p.19**
Master Method

\[ T_n = aT_{n/b} + f(n) \]

Case 2: \( f(n) \) “smaller” than \( n^{\log_b a} \)
Master Method

\[ T_n = aT_{n/b} + f(n) \]

Case 3: \( f(n) \) “equal” \( n^{\log_b a} \)
Master Method Summary

Given
- \( T_n = aT_{n/b} + f(n) \)

Three general cases (in the long run):
- \( f(n) \) grows faster than \( n^{\log_b a} \)
  - \( T_n \) grows proportional to \( f(n) \)
- \( f(n) \) grows slower than \( n^{\log_b a} \)
  - \( T_n \) grows proportional to \( n^{\log_k c} \)
- \( f(n) \) grows proportional to \( n^{\log_b a} \)
  - \( T_n \) grows proportional to \( \log_b n \ n^{\log_b a} \)
Master Method

Restrictions

- “Smaller” and “larger” mean polynomially smaller and larger
  - Must be some factor of $n^\epsilon$ different

- “Equal” means within a constant factor
- “Larger” also requires $f(n)$ is regular
  - As long as $f(n)$ is polynomial, it’s regular

$f(n)$ may be in none of the three cases!
Examples of Master Method

\[ T_n = 3T_{n/2} + n^2 \]

\[ T_n = 3T_{n/2} + n^{1.4} \]

\[ T_n = 16T_{n/4} + 10n^2 + 3n + 5 \]
Recurrence Relations

Linear homogenous recurrence relations with constant coefficients of order k

- Linear
- homogeneous
- constant coefficients
- order k
First-order example

$T_n = 3T_{n-1}$
Simple second-order example

\[ T_n = 4T_{n-2} \]

- Guess that there’s an exponential solution: \( T_n = r^n \)
Simple second-order example

\[ T_n = 4T_{n-2} \]

\[ T_0 = 1 \]

\[ T_1 = 3 \]

- Find particular solution of recurrence given initial conditions
- \[ T_n = C2^n + D(-2)^n \]
General Approach

Given a recurrence relation of form

- \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r} \)  \hspace{1cm} \text{linear homogeneous with constant coefficients}

Rewrite as powers:

- \( \alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \ldots + c_r \alpha^{n-r} \)

Simplify

- \( \alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{r-2} - \ldots - c_r = 0 \)  \hspace{1cm} \text{characteristic equation}

Solve for \( r \) distinct roots, \( \alpha_1, \ldots, \alpha_r \)

Generate form of final solution (\( a_n = \alpha_i^n \) is a solution to rec. relation)

- \( a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \ldots + A_r \alpha_r^n \)

Use initial conditions to create a set of simultaneous equations

- \( a_0 = A_1 \alpha_1^0 + A_2 \alpha_2^0 + \ldots + A_r \alpha_r^0 \)
- \( \ldots \)
- \( a_{r-1} = A_1 \alpha_1^{r-1} + A_2 \alpha_2^{r-1} + \ldots + A_r \alpha_r^{r-1} \)

Solve for \( A_1, \ldots, A_r \)
Example

Find a closed form for the number of ways to make a stack of $n$ chips using red, white and blue chips and so that no two red chips are together

- $a_n =$
- $a_0 =$
- $a_1 =$
Example

Create a recursive formula to specify how many ways to climb an \( n \)-stair staircase if each step covers either one or two stairsteps.
Example

A boy has $n$ cents. Each day, he can buy one of:
- a candy bar (for 20 cents)
- a piece of gum (for 1 cent)
- an apple (for 10 cents)

How many ways are there he can buy items until the money is gone?
Example

How many $n$-digit ternary sequences without any occurrence of the subsequence 012?
Adjustment for Multiple Roots

Given a recurrence relation of form

- $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ra_{n-r}$

... 

Solve for $r$ roots, $\alpha_1, \ldots, \alpha_r$

If we have a root $\alpha_i$ of multiplicity 2, general form of equation is

- $a_n = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_i\alpha_i^n + A_{i+1}n\alpha_i^n + \ldots + A_r\alpha_r^n$
Example with Multiple Roots

Given
- \( a_n = 4a_{n-1} - 4a_{n-2} \)
- \( a_0 = 1 \)
- \( a_1 = 3 \)

Find a closed form for \( a_n \)