Chapter 14

MCMC:
Markov Chain Monte Carlo
Why MCMC works:  
The structure of the proof

- We want to show that MCMC leads to *consistent* probability estimates.
- To do that, we:
  1. Note that (ergodic) Markov Chains reach just **one** stationary distribution $\Pi(x)$.
  2. Prove that **detailed balance** implies a stationary distribution.
  3. Prove that $\Pi(x) = P(x|e)$ and the Gibbs sampler satisfies detailed balance, and so $P(x|e)$ must be the stationary distribution that is reached.
Why MCMC works: The structure of the proof

1. Note that (ergodic) Markov Chains reach just one stationary distribution $\Pi(x)$.

(Ergodic just means all possible states are reached.)

Recall the probability of being in state $x'$ at time $t+1$ for a Markov Chain is:

$$\pi_{t+1}(x') = \sum_x \pi_t(x) \cdot q(x \rightarrow x')$$

A stationary distribution means remove “t”:

$$\pi(x') = \sum_x \pi(x) \cdot q(x \rightarrow x') \quad \forall x'$$
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2. Prove that \textit{detailed balance} implies a stationary distribution:

Detailed balance is more restrictive than stationary:

\[
\pi(x) \cdot q(x \rightarrow x') = \pi(x') \cdot q(x' \rightarrow x)
\]

So, plugging in to previous equation:

\[
\sum_x \pi(x) \cdot q(x \rightarrow x') = \sum_x \pi(x') \cdot q(x' \rightarrow x)
\]

\[
= \pi(x') \sum_x q(x' \rightarrow x) = \pi(x')
\]  

Q.E.D.
Why MCMC works:  
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Prove that $\Pi(x) = P(x|e)$ and the Gibbs sampler satisfies detailed balance, and so $P(x|e)$ must be the stationary distribution that is reached.

The Gibbs sampler is the simple idea of updating one variable at a time, which is what we have been doing:

$$q(x \rightarrow x') \triangleq q((x_i, \overline{x_i}) \rightarrow (x'_i, \overline{x_i})) = P(x'_i | \overline{x_i}, e)$$