1 Definitions

- The sample space, $S$ is the set of all possible outcomes.
- Elementary events are points in the sample space.
- Events are subsets of the sample space.

For example, consider the possible outcomes of rolling a six-sided die:

Events are mutually exclusive if their sets of points are disjoint.

Consider the possible outcomes of rolling a red die and a blue die:

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2 Basic Facts

For arbitrary events $A, B \subseteq S$, the following hold:

- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- $P(A \cup B) = P(A) + P(B)$ if $A$ and $B$ are mutually independent or disjoint
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

3 Mutually Exclusive Events

$A$ and $B$ are disjoint if $P(A \cap B) = 0$

Consider once again the red die and the blue die.

Let $A$ be the event that the sum of the rolls is 3.
Let $B$ be the event that the sum of the rolls is 10.

Let $A$ be the event that the sum of the rolls is 3.
Let $B$ be the event that the sum of the rolls is 10.

$P(A) = \frac{2}{36}$ (see the table above and count the ways it can happen)
$P(B) = \frac{3}{36}$
$P(A \cap B) = P(\text{sum} = 3 \text{ and } \text{sum} = 10) = 0$

4 Conditional Probability

$P(A|B)$ is the probability of $A$ given $B$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) \neq 0$

For example, flip two coins (HH, HT, TH, TT). Let $A$ be the event that two heads are flipped, and let $B$ be the event that at least one head is flipped. What is the probability that two heads are flipped given that at least one head is flipped?

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$
5 Independent Events

Events $A$ and $B$ are independent if and only if $P(A \cap B) = P(A)P(B)$. For each example, are the events independent?

Example 1:
$A$: blue die = 1 or 2, $P(A) = 1/3$
$B$: red die = 1 or 2, $P(B) = 1/3$
$P(A \cap B) = 4/36 = 1/9$

Example 2:
$A$: blue die = 5, $P(A) = 1/6$
$B$: red die + blue die = 10, $P(B) = 3/36 = 1/12$
$P(A \cap B) =$?

Example 3:
$A$: red $\leq$ 3 and blue $\leq$ 3, $P(A) = 1/4$
$B$: red = 1 or 2, $P(B) = 1/3$
$P(A \cap B) =$?

For independent events,

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \cap B) = P(B \cap A) = P(A)P(B)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

Example:
$A$: blue die = 4, 5, or 6, $P(A) = 1/2$
$B$: red die + blue die = 10, $P(B) = 1/12$
$P(A \cap B) = 3/36 \neq P(A)P(B) = 1/2 \times 1/12 = 1/24$

If the blue die is 4, 5, or 6, the sum is more likely to be 10.