Iterative Integration

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In this writeup we describe a method to integrate the gradient of a 2D scalar field. This is a widely studied problem especially in the context of shape from shading. Horn [1] describes an iterative update rule in section 11.8. Read section 11.7 and 11.8 to get an insight into how the update rule can be derived.

We are given the partial derivatives p and q of a scalar field S. In the context of photometric stereo, S is the surface we want to reconstruct. Using three (or more) images under three (or more) lighting conditions, one can recover the surface normal $\mathbf{n}_{i,j}$ at each pixel (i,j) in the image. The gradient of the surface at a pixel is related to the surface normal at that pixel by $\mathbf{n} = [-p \;\; -q \;\; 1]^\top$.

We start with some initial surface $S_{i,j} = 0$ and iteratively update S according to an update equation which is derived from the partial derivatives p and q of S. In other words at each step S would be updated such that the error between the partial derivative S_x and S_y calculated from the current surface and the partial derivatives p and q is minimized. The reconstruction error $e_{i,j}$ at each pixel (i,j) is given by

$$e_{i,j} = \frac{1}{4}[(S_{i+1,j} - S_{i,j} - p_{i,j})^2 + (S_{i,j+1} - S_{i,j} - q_{i,j})^2] \tag{1}$$

(Since S_x can approximated by $S_{i+1,j} - S_{i,j}$ and S_y can be approximated by $S_{i,j+1} - S_{i,j}$). We would like to minimize this error over all pixels. The total error can now be written as

$$\mathcal{E} = \sum_{i,j} e_{i,j} \tag{2}$$

It is possible that the scalar field under consideration is not defined at all positions (i,j) in a grid. For example, in the case of photometric stereo the object we are imaging might not cover the entire image. We define a mask $\mathcal{M}$ such that $m_{i,j}$=1 if the pixel at location (i,j) is part of the object, $m_{i,j}$=0 otherwise. The way $S_{i,j}$ is updated depends on which neighbors of pixel (i,j) are part of the object.

Differentiating equation 2 wrt $S_{i,j}$ and setting the derivative to zero

$$\frac{\partial \mathcal{E}}{\partial S_{i,j}} = 0 \tag{3}$$

we get the following iterative update rule:

$$\begin{aligned} S^{n+1}_{i,j} &= \frac{m_{i+1,j}(S^n_{i+1,j} - p_{i,j}) + m_{i-1,j}(S^n_{i-1,j} + p_{i-1,j})}{m_{i+1,j} + m_{i-1,j} + m_{i,j+1} + m_{i,j-1}} \\ &+ \frac{m_{i,j+1}(S^n_{i,j+1} - q_{i,j}) + m_{i,j-1}(S^n_{i,j-1} + q_{i,j-1})}{m_{i+1,j} + m_{i-1,j} + m_{i,j+1} + m_{i,j-1}} \qquad (4) \\ S^0_{i,j} &= 0 \qquad (5) \end{aligned}$$

The superscript refers to the number of iterations.

References

[1] B.K.P. Horn, Robot Vision, MIT Press, Cambridge, 1987.