### Pumping Lemma

If $A$ is a regular language, then there is a no. $p$ (pumping length) where, if $s$ is any string in $A$ of length at least $p$, $s$ may be divided into three pieces $x,y,z$, $s = xyz$, such that all of the following hold:

1. for each $i \geq 0$, $xy^iz$ is in $A$
2. $|y| > 0$
3. $|xy| \leq p$

Condition 1 lets us "pump out" elements in $A$

Note that $x$ or $z$ can be $\varepsilon$, but $y$ cannot be (by 2)

(Without 2, the lemma is trivially true, with $y = \varepsilon$)

Condition 3 assures us we can make $x$ and $y$ small, if needed.

### Alternating Quantifiers in the Pumping Lemma

1. For each regular language $L$  
   \[(Choice\ of\ L)\]
2. There exists a pumping length $p$ for $L$  
   \[(given\ p\ by\ lemma)\]
3. For every string $s$ in $L$ of length $\geq p$  
   \[(Choice\ of\ s)\]
4. There exists $x, y, z$, with $s = xyz$, $|y| > 0$ and $|xy| \geq p$.  
   \[(lengths\ y\ and\ xy\ restricted)\]
5. For each $i \geq 0$, $xy^iz$ in $L$.  
   \[(Choose\ an\ i\ that\ leads\ to\ contradiction)\]
### Pumping Lemma Example

$L = \{0^n1^n | n \geq 0\}$ is not regular.

Suppose $L$ were regular. Then let $p$ be the pumping length given by the pumping lemma.

Let $s = 0^p1^p$ in $L$. Note that $|s| > p$, so $s = xyz$ with

1. for each $i \geq 0$, $xy^iz$ is in $L$
2. $|y| > 0$

*(Don't use condition 3)*

**Case 1:** $y = 0 \ldots 0$

Then $xyyz$ will have more 0’s than 1’s, so it cannot be in $L$, a contradiction.

**Case 2:** $y = 1 \ldots 1$

Then $xyyz$ will have more 1’s than 0’s, so it cannot be in $L$, a contradiction.

**Case 3:** $y = 0 \ldots 0 1 \ldots 1$

Then $xyyz$ will have 0’s and 1’s out of order, with some 1’s before 0’s, a contradiction.

Since these are all possible cases, we can conclude that $L$ is not regular.
Pumping Lemma Example, using 3.

L = \{ 0^n 1^n \mid n \geq 0 \} is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.
Let s = 0^p 1^p in L. Note that |s| > p, so s = xyz with
1. for each i \geq 0, xy^i z is in L
2. |y| > 0
3. |xy| \leq p

It must be the case that y = 0 \ldots 0, since xy is shorter than p.

But then xyyz will have more 0’s than 1’s, so it cannot be in L, a contradiction.

Proving Languages Non-Regular

1. Assume L is regular. Then the Pumping Lemma holds.

2. Let p be the pumping length for L given by the lemma.

3. Choose a string s in L of length \geq p.
(Note that not every string in L will work!)

4. Consider all cases s can be divided into x, y, z, s = xyz satisfying conditions 2 and 3.
   Show for each case that there is an i \geq 0 with xy^i z not in L.

5. This provides a contradiction to the assumption of the pumping lemma, and to L being regular.
**Examples: Showing Language L Non-Regular**

\[
L = \{w | w \text{ has an equal no of 0 and 1's}\} \quad (1.39 \ p.80)
\]

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let \( p \) be the pumping length for L given by the lemma.
3. We choose \( s = 0^p1^p \) (in L of length \( \geq p \)).
4. Consider all cases \( s \) can be divided into \( x, y, z \),
   \( s = xyz \), satisfying conditions 2 (\(|y| > 0\)) and condition 3 (\(|xy| < p\)).
   For this \( s \), it must be that \( y = 0^k \); this is the only case. We choose \( i=2 \): \( xy^2z \) will have more
   0’s than 1’s, and so cannot be in L.
5. Contradiction.

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**Examples: Showing Language L Non-Regular**

\[
L = \{ww | w \text{ in } \{0, 1\}^* \} \quad (1.40 \ p.81)
\]

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let \( p \) be the pumping length for L given by the lemma.
3. We choose \( s = 0^p1^p \) (in L of length \( \geq p \)).
4. Consider all cases \( s \) can be divided into \( x, y, z \),
   \( s = xyz \), satisfying conditions 2 (\(|y| > 0\)) and condition 3 (\(|xy| < p\)).
   For this \( s \), it must be that \( y = 0^k \); this is the only case. We choose \( i=2 \): \( xy^2z \) will have more
   0’s before the first 1 than the second 1; not in L.
5. Contradiction.
Examples: Showing Language L Non-Regular

\[ L = \{0^i1^j \mid i > j\} \]  

1. Assume \( L \) is regular. Then the Pumping Lemma holds.
2. Let \( p \) be the pumping length for \( L \) given by the lemma.
3. We choose \( s = 0^{p+1}1^p \) (in \( L \) of length \( \geq p \)).
4. Consider all cases \( s \) can be divided into \( x, y, z \), \( s = xyz \), satisfying conditions 2 (\(|y| > 0\)) and condition 3 (\(|xy| < p\)).
   For this \( s \), it must be that \( y = 0^k \); this is the only case. We choose \( i=0 \): \( xz \) does not have more 0’s than 1’s, and so cannot be in \( L \). \( (i = 2?) \)
5. Contradiction.

Another Example

Show \( \{ a^n b^n c^n \mid n \geq 0 \} \) is not regular.

Show \( \{ a^n b a^n b a^{n+m} \mid n, m \geq 1 \} \) is not regular.
**Pumping Lemma Proof**

Proof Idea: Let $M$ be a DFA recognizing $A$.

Let $p = \text{the number of states in } M$.

Consider any string $s$ in $A$ of length $\geq p$ (if none, done!)

Suppose $s = s_1 s_2 \ldots s_n$.

Consider the sequence of states $q_0 q_1 q_2 \ldots q_n$ go through on $s$.

$q_0 \rightarrow s_1 q_1 \rightarrow s_2 \rightarrow \ldots \rightarrow s_{n-1} q_{n-1} \rightarrow q_n$ (final)

There are $n + 1$ states, $n \geq p$.

But $p$ is the number of states of $M$, so there must be a repetition. (Pigeonhole principle) Pick the first one.

$q_0 \rightarrow s_1 \rightarrow s_{r-1} \rightarrow q_r \rightarrow \ldots \rightarrow q_{r+i-1} \rightarrow \ldots \rightarrow q_{n-1} \rightarrow q_n$

$x \quad y \quad z$

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**Proof of the Pumping Lemma**

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing $A$.

Let $p = |Q|$, the number of states of $M$.

Let $s = s_1 \ldots s_n$ be any string of length $n$, $n \geq p$.

Let $r_1 \ldots m+1$ be the sequence of states that $M$ enters when processing $s$, so $r_i+1 = \delta(r_i, s_i)$ for $1 \leq i \leq n$. In the first $p+1$ elements of this sequence $r_1 \ldots m+1$, there must be 2 states that are the same, by the pigeonhole principle.

Call them $r_j$ and $r_l$.

We let $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots s_{l-1}$, $z = s_l \ldots s_n$.

$M$ must accept $x y^i z$ for all $i > 0$

$j$ is not equal to $l$, so $|y| > 0$

$l$ is less than $p+1$, so $|xy| \leq p$, satisfying all cond.