Countable sets
A set $A$ is finite means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$.
A set $A$ is countably infinite means it is the same size as $\mathbb{N}$.
A set $A$ is countable means it is either finite or countably infinite.

Extra example Prove or disprove: There is a set $Y$, $\neg (|Y| = |Y \times Y|)$

Extra example Prove or disprove: There is a set $Y$, $\neg (|Y| = |\mathcal{P}(Y)|)$

$\mathbb{N}$ and its power set
Example elements of $\mathbb{N}$

Example elements of $\mathcal{P}(\mathbb{N})$ Recall: For set $A$, its power set is $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$
Claim: There is an uncountable set. Example: __________

Proof: By definition of countable, since __________ is not finite, to show is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$. 

Rewriting using the definition of cardinality, to show is

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: $f$ is not a bijection. It’s enough to show that $f$ is not onto.

Rewriting using the definition of onto, to show:

\[ \neg \forall B \in \mathcal{P}(\mathbb{N}) \exists a \in \mathbb{N} \ ( f(a) = B ) \]

. By logical equivalence, we can write this as an existential statement:

In search of a witness, define the following collection of nonnegative integers:

\[ D_f = \{ n \in \mathbb{N} \mid n \notin f(n) \} \]

. By definition of power set, since all elements of $D_f$ are in $\mathbb{N}$, $D_f \in \mathcal{P}(\mathbb{N})$. It’s enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N} \ ( f(a) \neq D_f )$.

Proof of lemma:

By the Lemma, we have proved that $f$ is not onto, and since $f$ was arbitrary, there are no onto functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. QED

Where does $D_f$ come from? The idea is to build a set that would “disagree” with each of the images of $f$ about some element.

<table>
<thead>
<tr>
<th>$n \in \mathbb{N}$</th>
<th>$f(n) = X_n$</th>
<th>Is $0 \in X_n$?</th>
<th>Is $1 \in X_n$?</th>
<th>Is $2 \in X_n$?</th>
<th>Is $3 \in X_n$?</th>
<th>Is $4 \in X_n$?</th>
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<th>Is $n \in D_f$?</th>
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