Image Formation:
Geometric Camera Models

Computer Vision I
CSE 252A
Lecture 2
Announcements

• Course website
  https://cseweb.ucsd.edu/classes/fa19/cse252A-a/
• Piazza and Gradescope
• Homework 1 will be assigned today
  – Python
  – Due Tue, Oct 8, 11:59 PM
• Wait list
• Reading:
  – Chapters 1: Geometric camera models
Earliest Surviving Photograph

- First photograph on record, “la table service” by Nicephore Niepce in 1822.
- Note: First photograph by Niepce was in 1816.
How Cameras Produce Images

• Basic process:
  – photons hit a detector
  – the detector becomes charged
  – the charge is read out as brightness

• Sensor types:
  – CCD (charge-coupled device)
    • high sensitivity
    • high power
    • cannot be individually addressed
    • blooming
  – CMOS
    • simple to fabricate (cheap)
    • lower sensitivity, lower power
    • can be individually addressed
Images are two-dimensional patterns of brightness values.

They are formed by the projection of 3D objects.
Effect of Lighting: Monet
Change of Viewpoint: Monet

Haystack at Chailly at sunrise (1865)
Image Formation: Outline

• Geometric camera models
• Light and shading
• Color
Pinhole Camera: Perspective projection

• Abstract camera model - box with a small hole in it
Camera Obscura

"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays". --- Leonardo Da Vinci

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)
Camera Obscura

- Used to observe eclipses (e.g., Bacon, 1214-1294)
- By artists (e.g., Vermeer).
Camera Obscura

Jetty at Margate England, 1898.

http://brightbytes.com/cosite/collection2.html (Jack and Beverly Wilgus)
A and C are same size, but A is further from camera, so its image A' is smaller.

(Forsyth & Ponce)
The projection of the point $P$ on the image plane $\Pi'$ is given by the point of intersection $P'$ of the ray defined by $PO$ with the plane $\Pi'$. 
Geometric properties of projection

- 3-D points map to **points**
- 3-D lines map to **lines**
- Planes map to **whole image** or half-plane
- Polygons map to **polygons**

- Important point to note: Angles & distances not preserved, nor are inequalities of angles & distances.
- Degenerate cases:
  - line through focal point projects to **point**
  - plane through focal point projects to a **line**
Equation of Perspective Projection

Cartesian coordinates:

- We have, by similar triangles, that for \( P = (x, y, z) \), the intersection of \( OP \) with \( \Pi' \) is \( (f' \frac{x}{z}, f' \frac{y}{z}, f') \)

- Establishing an image plane coordinate system at \( C' \) aligned with \( i \) and \( j \), we get \( (x, y, z) \rightarrow (f' \frac{x}{z}, f' \frac{y}{z}) \)
Virtual Image Plane

- Virtual image plane in front of optical center.
- Image is ‘upright’

\[(x, y, z) \rightarrow (f'' \frac{x}{z}, f''' \frac{y}{z})\]
A Digression

Projective Geometry
and
Homogenous Coordinates
What is the intersection of two lines in a plane?

A Point
Do two lines in the plane always intersect at a point?

No, Parallel lines don’t meet at a point.
Can the perspective image of two parallel lines meet at a point?

YES
Projective geometry provides an elegant means for handling these different situations in a unified way and homogenous coordinates are a way to represent entities (points & lines) in projective spaces.
Projective Geometry

• Axioms of Projective Plane
  1. Every two distinct points define a line
  2. Every two distinct lines define a point (intersect at a point)
  3. There exists three points, A, B, C such that C does not lie on the line defined by A and B.

• Different than Euclidean and affine geometry

• Projective plane is “bigger” than affine plane – includes “line at infinity”
Homogeneous coordinates

• Boardwork
  – 2D points and lines
  – Point at infinity
  – Line at infinity
Homogeneous coordinates

- 3D point using inhomogeneous coordinates as 3-vector
  \[ \tilde{X} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} \]

- 3D point using affine homogeneous coordinates as 4-vector
  \[ X = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \\ 1 \end{bmatrix} \]
Homogeneous coordinates

- 3D point using *affine* homogeneous coordinates as 4-vector

\[ \mathbf{x} = \begin{bmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \\ 1 \end{bmatrix} \]

- 3D point using *projective* homogeneous coordinates as 4-vector (*up to scale*)

\[ \mathbf{x} = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \]
Homogeneous coordinates

• Projective homogeneous 3D point to affine homogeneous 3D point

\[ \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \frac{1}{W} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} \]

• Dehomogenize 3D point

\[ \tilde{\mathbf{X}} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \end{bmatrix} \]
Homogeneous coordinates

- Homogeneous points are defined up to a nonzero scale

\[
\begin{align*}
X &= \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \lambda \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} \lambda X \\ \lambda Y \\ \lambda Z \\ \lambda W \end{bmatrix} \\
\tilde{X} &= \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} \frac{\lambda X}{\lambda W} \\ \frac{\lambda Y}{\lambda W} \\ \frac{\lambda Z}{\lambda W} \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \end{bmatrix}
\end{align*}
\]
Homogeneous coordinates

- When $W = 0$, then it is a point at infinity
- Affine homogeneous coordinates are projective homogeneous coordinates where $W = 1$
- When not differentiating between affine homogeneous coordinates and projective homogeneous coordinates, simply call them homogeneous coordinates
End of the Digression
In a perspective image, parallel lines meet at a point, called the vanishing point.

Doesn’t need to be near the center of the image.
Parallel lines meet in the image

- Vanishing point location: Intersection of 3-D line through $O$ parallel to given line(s)
- A single line can have a vanishing point
Vanishing points

Different directions correspond to different vanishing points
Vanishing Points
Vanishing Point

• In the **projective plane**, parallel lines meet at a point at infinity.

• The 2D vanishing point in the image is the perspective projection of this 3D point at infinity.
What is a Camera?

• An mathematical expression that relates points in 3D to points in an image for different types of physical cameras or imaging situations
Geometry

• How do 3D world points project to 2D image points?
The equation of projection

Homogenous coordinates and camera matrix

Cartesian coordinates:

$$(x, y, z) \rightarrow \left( f \frac{x}{z}, f \frac{y}{z} \right)$$
What if camera coordinate system differs from world coordinate system?

Camera coordinate frame

World coordinate frame
Special cases

- Imaging a plane
- Only camera rotation (no translation)
- In both cases, mapping between images is a planar projective transformation

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix} =
\begin{bmatrix}
  a_{11} & a_{12} & a_{21} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]
Planar Homography

\[ x = H_1 X \]
\[ x' = H_2 X \]
\[ x' = H_2X = H_2(H_1^{-1} x) = (H_2H_1^{-1})x \]

Application: Two photos of a white board
Planar Homography: Pure Rotation

\[ x' = H_2X = H_2(H_1^{-1}x) = (H_2H_1^{-1})x \]

Application: Panoramas
Application: Panoramas and image stitching

All images are warped to central image
Euclidean Coordinate Systems
Coordinate Change: Rotation Only

\[ X' = RX \]
Coordinate Change: Translation Only

\[ X' = X + t \]
Coordinate Changes: Rotation and Translation

\[ X' = RX + t \]
Some points about SO(n)

- \( \text{SO}(n) = \{ R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1 \} \)
  - \( \text{SO}(2) \): rotation matrices in plane \( \mathbb{R}^2 \)
  - \( \text{SO}(3) \): rotation matrices in 3-space \( \mathbb{R}^3 \)

- Forms a Group under matrix product operation:
  - Identity
  - Inverse
  - Associative
  - Closure

- Closed (finite intersection of closed sets)
- Bounded \( R_{i,j} \in [-1, +1] \)
- Does not form a vector space.
- Manifold of dimension \( n(n-1)/2 \)
  - \( \dim(\text{SO}(2)) = 1 \)
  - \( \dim(\text{SO}(3)) = 3 \)
Parameterizations of SO(3)

– Even though a rotation matrix is 3x3 with nine numbers, it only has three degrees of freedom. It can be parameterized with three numbers. There are many parameterizations.

• Other common parameterizations
  – Euler Angles
  – Axis Angle
  – Quaternions
    • four parameters; homogeneous
Rotation: Homogenous Coordinates

• About z axis

\[
\begin{pmatrix}
x' \\
y' \\
z' \\
1
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix}
\]
Rotation

- About x axis:

\[
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
\]

- About y axis:

\[
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
\]
Euler Angles: Roll-Pitch-Yaw

- Composition of rotations
  \[
  R = R_Z(\gamma) R_Y(\beta) R_X(\alpha)
  \]

  \[
  R = \begin{bmatrix}
  \cos \gamma & -\sin \gamma & 0 \\
  \sin \gamma & \cos \gamma & 0 \\
  0 & 0 & 1 
  \end{bmatrix}
  \begin{bmatrix}
  \cos \beta & 0 & \sin \beta \\
  0 & 1 & 0 \\
  -\sin \beta & 0 & \cos \beta 
  \end{bmatrix}
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \alpha & -\sin \alpha \\
  0 & \sin \alpha & \cos \alpha 
  \end{bmatrix}
  \]
What if camera coordinate system differs from world coordinate system?

\[ X_{\text{Camera}} = R X_{\text{World}} + t \]

Camera coordinate frame

World coordinate frame
Intrinsic parameters

- 3x3 homogenous matrix
- Focal length
- Principal Point
- Units (e.g. pixels)
- Pixel Aspect ratio
Given $n$ points $P_1, \ldots, P_n$ with known positions and their images $p_1, \ldots, p_n$, estimate intrinsic and extrinsic camera parameters.

- See Textbook for how to do it.
- Camera Calibration Toolbox for Matlab (Bouguet)
  - http://www.vision.caltech.edu/bouguetj/calib_doc/
Camera parameters

• Extrinsic Parameters: Since camera may not be at the origin, there is a rigid transformation between the world coordinates and the camera coordinates.

• Intrinsic parameters: Since scene units (e.g., cm) differ image units (e.g., pixels) and coordinate system may not be centered in image, we capture that with a 3x3 transformation comprised of focal length, principal point, pixel aspect ratio, and skew.

\[
\begin{pmatrix}
x \\ y \\ w
\end{pmatrix} = \begin{pmatrix}
\text{Transformation} & \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} & \text{Rigid Transformation} & \begin{pmatrix}
X \\ Y \\ Z \\ T
\end{pmatrix}
\end{pmatrix}

\text{3 x 3}

\text{represented by}

\text{extrinsic parameters}

\text{4 x 4}
Camera Models

- Perspective Projection
  - Affine Camera Model
    - Scaled Orthographic Projection
      - Orthographic Projection
For all cameras?
Other camera models

- Generalized camera – maps points lying on rays and maps them to points on the image plane.

Omnicam (hemispherical)  Light Probe (spherical)
Some Alternative “Cameras”
Lenses
Beyond the pinhole Camera
Getting more light – Bigger Aperture
Pinhole Camera Images with Variable Aperture

2 mm  1 mm

.6 mm  .35 mm

.15 mm  .07 mm
The reason for lenses
We need light, but big pinholes cause blur.
Thin Lens

- Rotationally symmetric about optical axis.
- Spherical interfaces.
Thin Lens: Center

- All rays that enter lens along line pointing at O emerge in same direction.
Thin Lens: Focus

Parallel lines pass through the focus, F
Thin Lens: Image of Point

- All rays passing through lens and starting at $P$ converge upon $P'$
- So light gather capability of lens is given the area of the lens and all the rays focus on $P'$ instead of become blurred like a pinhole
Thin Lens: Image of Point

Relation between depth of Point (-Z) and the depth where it focuses (Z')

\[
\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}
\]
A price: Whereas the image of $P$ is in focus, the image of $Q$ isn’t.
Thin Lens: Aperture

- Smaller Aperture -> Less Blur
- Pinhole -> No Blur
Next Lecture

• Image Formation: Light and Shading
• Reading:
  – Chapter 2: Light and Shading