• This is a closed book, closed notes exam. Switch off your cell phone and do not communicate with anyone other than an exam proctor.
• Start writing when instructed. Stop writing when your time is up.
• Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

(1) (5 Points) Suppose we have two random variables \(X\) and \(Y\) with the following joint distribution:

<table>
<thead>
<tr>
<th></th>
<th>(X = 1)</th>
<th>(X = 2)</th>
<th>(X = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y = 0)</td>
<td>1/3</td>
<td>1/4</td>
<td>1/6</td>
</tr>
<tr>
<td>(Y = 1)</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
</tr>
</tbody>
</table>

Write down the conditional distribution of \(X\) given \(Y = 1\).

Solution: Recall that the conditional probabilities can be calculated from the equation:

\[
\Pr(X = x|Y = y) = \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)}
\]

Here, \(\Pr(Y = 1) = \Pr(X = 1, Y = 1) + \Pr(X = 2, Y = 1) + \Pr(X = 3, Y = 1) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}\).

Therefore, the conditional distribution of \(X\) given \(Y = 1\) is described as:

\[
\begin{align*}
\Pr(X = 1|Y = 1) &= \frac{1/12}{1/4} = \frac{1}{3}, \\
\Pr(X = 2|Y = 1) &= \frac{1/12}{1/4} = \frac{1}{3}, \\
\Pr(X = 3|Y = 1) &= \frac{1/12}{1/4} = \frac{1}{3}.
\end{align*}
\]

(2) (5 Points) We roll two standard 6-sided dice. Let \(X\) represent the value of the first dice and \(Y\) the value of the second. Let random variable \(A = X + Y\) be the sum of their values. Let random variable \(B = X \cdot Y\) be the product of their values. What is the expected value of the sum of \(A\) and \(B\), i.e. \(E[A + B]\)? Please show your work. [Hint: Remember, \(X\) and \(Y\) are independent!]

Solution:

\[
E[A + B] = E[X + Y + XY]
\]

(expand variables)

\[
= E[X] + E[Y] + E[XY]
\]

(linearity of expectation)

\[
\]

(independence of \(X\) and \(Y\))

\[
= \frac{7}{2} + \frac{7}{2} + \frac{7}{2} \times \frac{7}{2}
\]

(plug in known values)

\[
= 19 \frac{1}{4}
\]