ECE 259A: Midterm Exam

Instructions: There are four problems, weighted as shown below. Notice that the maximum possible score on this exam is 110. This means that you have 10 bonus points. The exam is open book and open notes: you may use any auxiliary material that you like as long as it is on paper.

Good luck!

Problem 1. (15 points)
Let \( C \) be the binary linear code of length 6 defined by the following parity-check matrix:

\[
H = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

a. For this parity-check matrix \( H \), write-down a table of coset leaders and syndromes (of the corresponding cosets). The table should have eight rows, with each row consisting of the pair \{coset leader, syndrome\} in the following format: \{(000000), (000)\}. For cosets wherein the vector of minimum weight is not unique, you can select any such vector as the coset leader.

b. Suppose \( C \) is used to communicate over the binary symmetric channel BSC\( (p) \) with cross-over probability \( p < 0.5 \). Assuming all codewords are equally likely to be transmitted, express the probability of block error \( P_e \) under maximum-likelihood decoding in terms of \( p \).

Problem 2. (35 points)
Let \( C_1 \) be an \((n_1, k_1, d_1)\) binary linear code and let \( C_2 \) be an \((n_2, k_2, d_2)\) binary linear code. Consider the binary code \( C \) of length \( n_1 n_2 \) whose codewords are all the \( n_1 \times n_2 \) binary arrays \( X = [x_{ij}] \) with the following properties:

**P1.** Each row of \( X \) is a codeword of \( C_2 \). That is \((x_{i1}, x_{i2}, \ldots, x_{in_2}) \in C_2 \) for all \( i = 1, 2, \ldots, n_1 \).

**P2.** Each column of \( X \) is a codeword of \( C_1 \). That is \((x_{1j}, x_{2j}, \ldots, x_{nj}) \in C_1 \) for all \( j = 1, 2, \ldots, n_2 \).

a. Prove that \( C \) is a linear code and determine its dimension.

b. What is the minimum distance \( d \) of \( C \)? Prove your answer.

c. Suppose you have access to systematic encoders for \( C_1 \) and \( C_2 \). Using these encoders, design a systematic encoder for \( C \). Explain in detail how your encoder works and why.

d. The code \( C \) is used on the binary erasure channel that either transmits each bit as is or replaces it with the erasure symbol \( \phi \). Recall that a binary linear code \( C \) with distance \( d \) can correct any pattern of up to \( d - 1 \) erasures. Suppose you have access to decoders for \( C_1 \) and \( C_2 \) that correct up to \( d_1 - 1 \) and \( d_2 - 1 \) erasures, respectively. Use these decoders to design a decoder for \( C \) that corrects any pattern of up to \( d - 1 \) erasures, where \( d \) is the minimum distance of \( C \) from part (b).
Problem 3. (25 points)

Let $\mathcal{E}$ be an arbitrary subset of $\mathbb{F}_2^n$ of size $|\mathcal{E}| = M$, such that $0 \in \mathcal{E}$. We say that a code $C \subset \mathbb{F}_2^n$ detects all error patterns in the set $\mathcal{E}$ if $\xi_1 + \xi \neq \xi_2$ for all $\xi \in \mathcal{E}$ and for all distinct $\xi_1, \xi_2 \in C$ (or, equivalently, if $(\xi + \mathcal{E}) \cap C = \{\xi\}$ for all $\xi \in C$).

Show that there exist codes of cardinality $|C| \geq 2^n / M$ that detect all error patterns in the set $\mathcal{E}$.

Problem 4. (35 points)

Let $V$ and $A$ be subsets of $\mathbb{F}_2^8$ of size $|V| = |A| = 16$. It is known that $V \cap A = \{0\}$. It is also known that for any distinct $v_1, v_2 \in V$ and any distinct $a_1, a_2 \in A$, we have

$$v_1 + v_2 \neq a_1 + a_2 \quad (\ast)$$

Let $H$ be the $8 \times 15$ matrix whose columns are the 15 nonzero elements of $V$, listed in some fixed order. You may assume that the rows of $H$ are linearly independent. We define

$$C = \{ x \in \mathbb{F}_2^{15} : Hx^t \in A \}$$

Thus $C$ is a binary code of length 15. The problem is concerned with the properties of this code.

a. What is the number of codewords in $C$?

b. Prove that the distance between any two codewords of $C$ is at least 3.

c. What is the covering radius of $C$?