ECE 259A: Midterm Exam

Instructions: There are four problems, weighted as shown below. Notice that the maximum possible score on this exam is 110. This means that you have 10 bonus points. The exam is open book and open notes: you may use any auxiliary material that you like.

Good luck!

Problem 1. (15 points)

Let $C$ be a binary code of length $n$ with minimum distance $d$. A decoder for $C$ is a function $D: \mathbb{F}_2^n \rightarrow C$. Let $t = \lceil (d-1)/2 \rceil$. Show that $t$ is the maximum number of errors that any decoder for $C$ can guarantee to correct. That is, prove that for every decoder $D: \mathbb{F}_2^n \rightarrow C$, there exist a codeword $x$ and an error vector $e$ of weight $\leq t + 1$ such that $D(x + e) \neq x$.

Problem 2. (45 points)

Let $C_1$ and $C_2$ be two binary linear codes of the same length $n$. Denote $k_1 = \dim C_1$ and $k_2 = \dim C_2$. Let $d_1, d_2$ be the minimum distances of $C_1$ and $C_2$, respectively. Define the code $C_1 \otimes C_2$ as follows:

$$C_1 \otimes C_2 \overset{\text{def}}{=} \left\{ (x_1 | x_1 + x_2) : x_1 \in C_1 \text{ and } x_2 \in C_2 \right\}$$

a. Let $G_1$ and $G_2$ be generator matrices of $C_1$ and $C_2$, respectively, and let $O$ denote the $k_2 \times n$ all-zero matrix. Show that $C_1 \otimes C_2$ is a binary linear code generated by

$$G = \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix}$$

b. Let $H_1$ and $H_2$ be parity-check matrices for $C_1$ and $C_2$, and let $O$ denote the $(n-k_1) \times n$ all-zero matrix. Show that a parity-check matrix for $C_1 \otimes C_2$ is given by:

$$H = \begin{pmatrix} H_1 & O \\ H_2 & H_2 \end{pmatrix}$$

c. Prove that the minimum distance of $C_1 \otimes C_2$ is given by $d = \min\{2d_1, d_2\}$.

d. Now let $C_1$ be the binary linear code of length 2 given by $C_1 = \{00, 01, 10, 11\}$. For $m \geq 2$, define the code $C_m$ recursively as follows:

$$C_m \overset{\text{def}}{=} C_{m-1} \otimes \{0, 1\}$$

where 0, 1 denote the all-zero and all-one vectors of the appropriate length. Determine the parameters $n_m, k_m, d_m$ of the code $C_m$ for all $m \geq 1$. Can you recognize the code $C_3$?

e. Prove that the minimum distance of $C_m$ is the highest possible for its length and dimension.
Problem 3. (25 points)
A binary vector $e$ is said to be a burst of length $\ell$ if the difference between the last (rightmost) and the first (leftmost) nonzero positions in $e$ is exactly $\ell - 1$. For example, the following vectors:

\[(001000), (000011), (010100), (001101), (101110), (100001)\]

are bursts of length 1, 2, 3, 4, 5, 6, respectively. Let $C$ be an $(n,k)$ binary linear code, and suppose there exists a decoder for $C$ that corrects all bursts of length $t$ or less, where $t \geq 2$. Prove that

$$2^{n-k} \geq 1 + n + \sum_{\ell=2}^{t} (n-\ell+1)2^{\ell-2}$$

Problem 4. (25 points)
A binary linear code $C$ with parameters $n = 24$, $k = 14$, and $d = 6$ was discovered by T.J. Wagner. Show that the $(24,12,8)$ extended binary Golay code $C_{24}$ cannot be a subcode of $C$. 