Reminders: Midterm in 1 week.

Review session on Sunday.

CSE 20
DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Distinguish between a theorem, an axiom, lemma, a corollary, and a conjecture.
- Recognize direct proofs
- Recognize proofs by contraposition
- Recognize proofs by contradiction
- Recognize fallacious “proofs”
- Evaluate which proof technique(s) is appropriate for a given proposition
  - Direct proof
  - Proofs by contraposition
  - **Proofs by contradiction**
  - **Proof by cases**
  - **Constructive existence proofs**
- Correctly prove statements using appropriate style conventions, guiding text, notation, and terminology

Textbook references: Sections 1.7-1.8 (with reference to 1.6)

Overall strategy

- Do you believe the statement?
  - Try some small examples.
- Determine logical structure + main connective.
- Determine relevant definitions.
- Map out possible proof strategies.
  - For each strategy: what can we **assume**, what is the **goal**?
  - Start with simplest, move to more complicated if/when get stuck.

Proof strategies so far

Assume? Goal?

"Let"

- To prove a statement of the form All x have property P(x)   WTS
  - Consider an arbitrary (fixed but unknown) x in the domain. Prove P(x) holds for that element – only using facts true about "generic" elements in the domain.
- To prove a statement All x have property P(x) is false
  - Find a counterexample: a specific element in the domain where P(x) evaluates to False.
- To prove a statement of the form There is an x with property P(x)
  - Find an example: a specific element in the domain where P(x) evaluates to True.
- To prove a statement There is an x with property P(x) is false
  - Consider an arbitrary (fixed but unknown) x in the domain. Prove P(x) **fails**for that element – only using facts true about "generic" elements in the domain.

Proof strategies so far

Assume? Goal?

- To prove a statement of the form If P then Q
  - Assume P is true. Using this assumption (and definitions, etc.) prove Q.
  - OR Assume Q is false. Using this assignment (and defintiions, etc.) prove P is false.
- To prove a statement of the form Both X and Y
  - Step 1: Prove X. Step 2: Prove Y
- To prove a statement of the form At least one of X and Y
  - Convert to equivalent version: If not X then Y
  - OR convert to different equivalent version: If not Y then X

$X \vee Y \equiv (\neg X) \to Y \equiv (\neg Y) \to X$

What if we're completely stuck?

- Example: the square root of 2.

Which of the following is **not** true about $\sqrt{2}$?

A. It is positive.

B. It is an integer.

C. It is a real number.

D. It is less than 2.

E. It is greater than or equal to 1.

$1 < 2 < 4$

$\sqrt{1} = 1$ $\sqrt{4} = 2$

What if we're completely stuck?

- **Definition** (p. 85) A real number r is **rational** if there are integers p,q with q nonzero such that r = p/q

Is the square root of 2 rational? i.e. is the stmt $\exists p \exists q (\frac{p}{q} = \sqrt{2}) \wedge q \neq 0)$ True?

A. Yes, because it is between 1 and 2.

B. Yes, because it can be written as the result of an operation (square root) applied to a rational number (2).

C. Yes, for a different reason.

D. No, because when we write it as the fraction $\sqrt{2}/1$, the numerator isn't an integer.

E. No, for a different reason.

Proving a number is not rational

- **Goal:** $\sqrt{2}$ is not rational.
- Rephrasing the **goal** : It is not possible to find integers p,q (with q nonzero) such that $(p/q)^2 = 2$.
- Proof ? WTS $\neg \exists p \exists q \left(q \neq 0 \wedge \left(\frac{p}{q}\right)^2 = 2 \right)$

$$\equiv \forall p \forall q \neg \left(q \neq 0 \wedge \left(\frac{p}{q}\right)^2 = 2 \right)$$

$$\equiv \forall p \forall q \left(q \neq 0 \rightarrow \left(\frac{p}{q}\right)^2 \neq 2 \right)$$

Proof by contradiction

Assume? Goal?

- To prove a statement of **any form**
  - Assume: the statement is false.
  - Using this assumption (and definitions, etc.) find X which must be both truth and false!
  - ????
  - Conclude: original assumption invalid, i.e. the statement is true.

Proving a number is not rational

- **Goal:** $\sqrt{2}$ is not rational.

- Rephrasing the **goal** : It is not possible to find integers p,q (with q nonzero) such that $(p/q)^2 = 2$.

- Proof by contradiction:
  - Assume we have integers p,q (with q nonzero) such that $(p/q)^2 = 2$.
  - ….

Another example

Rosen 1.7 p. 91 #24

- Show that at least three of any 25 days chosen must fall in the same month of the year.

$25 = 2 \cdot 12 + 1$

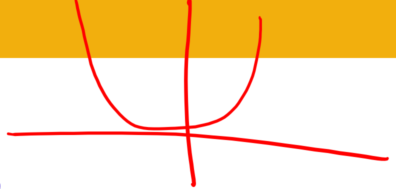
Domain: days that are picked.

$\exists x \exists y \exists z (\quad)$

distributing . . .

Pf by contradiction: Assume each month has ≤ 2 of selected days. There are 12 months, there are ≤ 24 selected days $\rightarrow\leftarrow$

Proof by cases



Rosen 1.8 p. 93 Example 3

$\geq$

- For every integer n, n^2 is greater than or equal to n.

Thm: $\forall n \; (n^2 \geq n)$

Alt Pf: TWA ~~x~~, assume $\exists n \; (n^2 < n)$? ? but $1^2 = 1$ . . . — . . ? ?

Pf: Let n be an int.

Case ① $n > 0$ so $n \geq 1$. . Multiply by n (pos)
$n^2 \geq n$:)

Case ② $n = 0$. So $n^2 = 0 = n$ so $n^2 \geq n$:)

Case ③ $n < 0$ neg n^2 pos b/c $n \neq 0$. so $n < n^2$ i.e. $n \leq n^2$:)

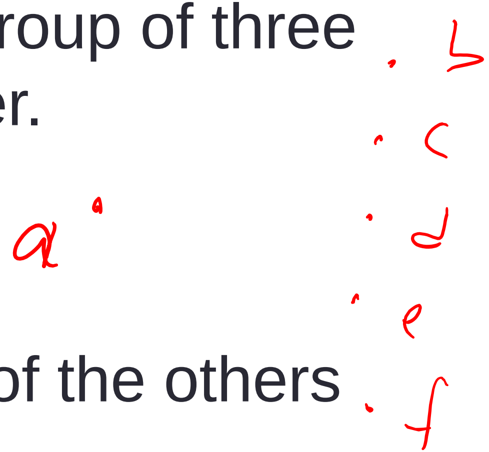
Another example

- Every collection of 6 people includes a group of three people who all know one another or a group of three people who are strangers to one another.

a, b, c, d, e, f.

Proof by (hidden) cases

- Every collection of 6 people includes a group of three people who all know one another or a group of three people who are strangers to one another.



IDEA: Focus on one person

- Either that person knows at least three of the others

- Or that person knows at most two of the others.

Proof by (hidden) cases

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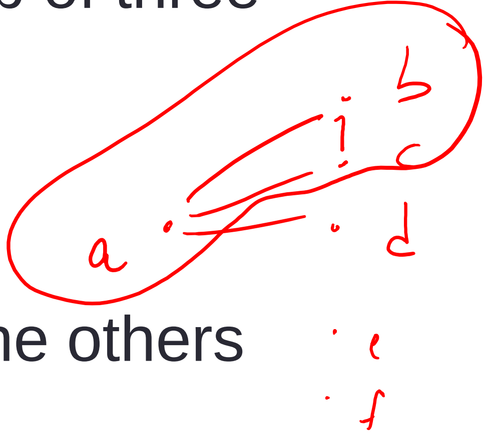


IDEA: Focus on one person

- Either that person knows at least three of the others
  - Among these three, do any know one another?
- Or that person knows at most two of the others.
  - Among unknown people, are there any strangers?

Proof by (hidden) cases

Thm: In every graph w/ 6 nodes, there is 3-clique or there are 3 nodes w/ no edges between them.

- Every collection of 6 people includes a group of three people who all know one another or a group of three people who are strangers to one another.

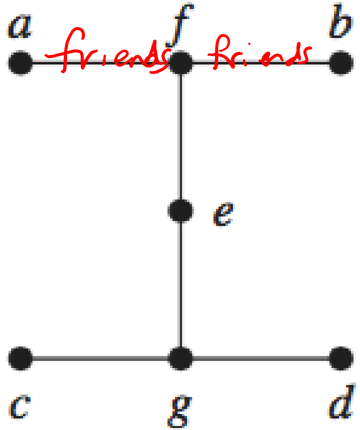
(simple)

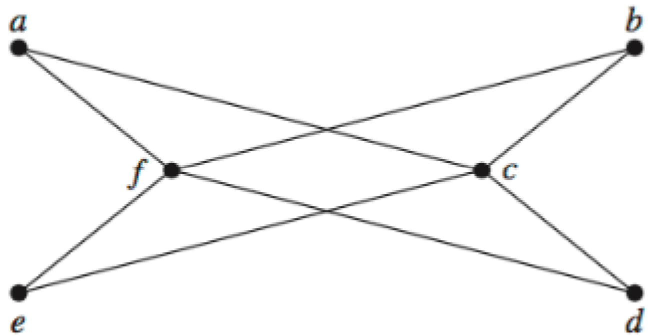
Graphs!

nodes

edges

a, b stranger





Existence proofs

Rosen 1.8 p. 97 Example 11

- Can we choose two irrational numbers x and y where x^y is rational?

Overall strategy

- Do you believe the statement? its negation?
  - Try some small examples.
- Determine logical structure + main connective.
- Determine relevant definitions.
- Map out possible proof strategies.
  - For each strategy: what can we **assume**, what is the **goal**?
  - Start with simplest, move to more complicated if/when get stuck.

Direct proof, construction, exhaustive, etc.

Contradiction, hidden cases