# CSE 20
# DISCRETE MATH

Note: Sub on Thursday

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

# Today's learning goals

- Determine the truth value of predicates for specific values of their arguments
- Determine the truth sets of predicates
- Define the universal and existential quantifiers
- Translate sentences from English to predicate logic using appropriate predicates and quantifiers
- Use appropriate Boolean operators to restrict the domain of a quantified statement
- Negate quantified expressions
- Translate quantified statements to English, even in the presence of nested quantifiers
- Evaluate the truth value of a quantified statement with nested quantifiers

# Predicates and Quantifiers

*Rosen p. 37-44*

**Predicate:** *informally*, a proposition with a "hole"

P(x) is "x > 3"

Q(x) is "the word x contains the letter 'a'"

Domain / Universe i.e. type

**x has different possible values depending on predicate**

# Predicates and Quantifiers

$\pm\sqrt{b^2 - 4ac}$ ...

Consider the predicate P(x) is "$x^2 - 4 = 0$"

solving: $x^2 = 4$ so $x = \pm 2$

Is there some value of x which makes P(x) true?

A. Yes, all integer values of x make P(x) true.

Counterex $x = 1, 3, 5, 19, \ldots$ (only need one)

B. Yes, there's exactly one positive integer value of x that make this predicate evaluate to T.

C. Yes, there are exactly two integer values of x that make this predicate evaluate to T.

D. No, there are no values in {-1,0,1} which make P(x) true.

E. More than one of the above.

Universe matters!

# Universal quantifiers

"P(x) for all values x in the domain"

$$\forall x P(x)$$

$\forall x P(x)$ is T when P(x) is "$\log_2 x < x$" and the domain is integers greater than 1.

$\forall x P(x)$ is F when P(x) "$x^2 > x$" and the domain is all real numbers.

# Counterexample

"P(x) for all values x in the domain"

$$\forall x P(x)$$

To **disprove** a universal statement: give a counterexample.

- element in the domain
- which makes the predicate F.

Example: proving two compound propositions are **not** logically equivalent

# Existential quantifiers

"There exists an element in the domain such that P(x)"

$$\exists x P(x)$$

$\exists x P(x)$ is T when P(x) is "$x^2 > x$" and the domain is all real numbers.

$\exists x P(x)$ is F when P(x) is "$x^2 + 1 = 0$" and the domain is all real numbers.

# Construction proof

"There exists an element in the domain such that P(x)"

$$\exists x P(x)$$

To **prove** an existential statement: give an example.

- element in the domain
- which makes the predicate T.

Example: proving several compound propositions **are** consistent

# "De Morgan"-ish

*Rosen p. 45*

A universal claim is False

$$\neg \forall x P(x) \equiv \exists x\, (\neg P(x))$$

counterexample ... the existential claim is true

example where P(x) is F.

$$\neg \exists x P(x) \equiv \forall x\, (\neg P(x))$$

no example — check each candidate example fails

**Logical equivalence of quantified statements**: no matter which predicates are substituted and no matter which domains of discourse, the statements have the same truth value

# Restricting the domain

*Rosen p. 44*

Over the domain of real numbers,

$$\forall x > 1 (x^2 > x)$$

means

$$\forall x (x > 1 \rightarrow x^2 > x)$$

"Every real number greater than 1 makes $x^2 > x$ True."

# Restricting the domain

*Rosen p. 44*

Over the domain of real numbers,

$\forall x > 1(x^2 > x)$ means $\forall x(x > 1 \rightarrow x^2 > x)$

"Every real number greater than 1 makes $x^2 > x$ T."

"There is a real number greater than 1 ~~,~~ and that makes P(x) true"

A. Translates as $\exists x(x > 1 \rightarrow x^2 > x)$ tempting b/c can find ex where $\exists x(x>1 \rightarrow \ldots)$ T but isn't pf of statement e.g. 0.5

B. Translates as $\exists x(x > 1 \vee x^2 > x)$

C. Translates as $\exists x(x > 1 \wedge x^2 > x)$

D. Translates as $\exists x(x > 1 \leftrightarrow x^2 > x)$ tempting

# Restricting the domain

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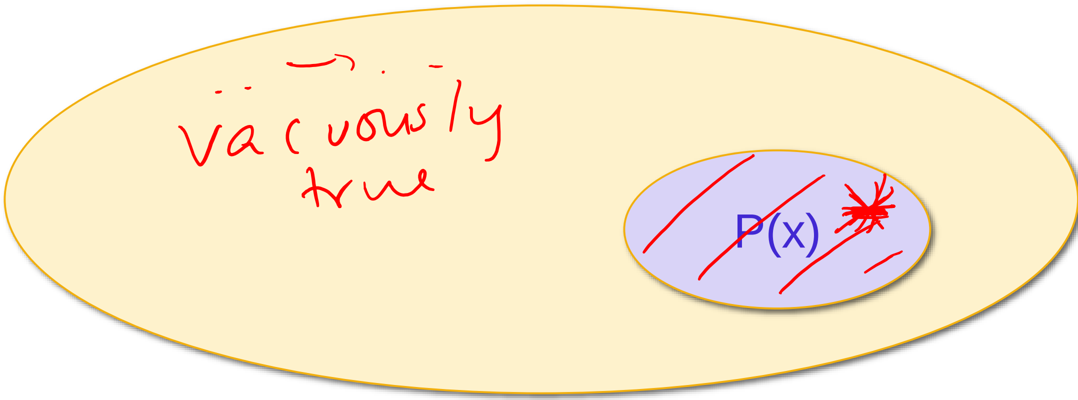
Over the domain of real numbers,

$\forall x > 1 (x^2 > x)$ means $\forall x (x > 1 \rightarrow x^2 > x)$

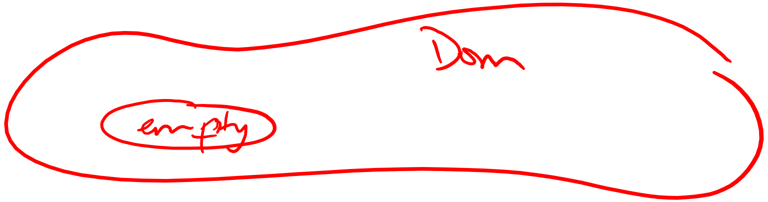
"Every real number greater than 1 makes $x^2 > x$ T."

$$\forall x (P(x) \rightarrow Q(x))$$

$$\exists x (P(x) \wedge Q(x))$$



# Edge case: empty domain?




$\forall x\,(E(x) \to P(x))$

$F$

What if the domain of discourse is empty?

A. $\forall x P(x)$, $\exists x P(x)$ both evaluate to T.
B. $\forall x P(x)$ evaluates to T and $\exists x P(x)$ evaluates to F.
C. $\forall x P(x)$ evaluates to F and $\exists x P(x)$ evaluates to T.
D. $\forall x P(x)$, $\exists x P(x)$ both evaluate to F.
E. Quantifiers are not defined in this case.

Domain ?

# Translations

*Rosen p. 54 #28*

Something is not in the correct place.

Everything is in the correct place and in excellent condition.

All tools are in the correct place and are in excellent condition.

Nothing is both in the correct place and is in excellent condition.

One of the tools is not in the correct place, but it is in excellent condition.

# Translations

*Rosen p. 54 #28*

Something is not in the correct place.

$$\exists x(\neg P(x))$$

Everything is in the correct place and in excellent condition.

$$\forall x(P(x) \wedge D(x))$$

All tools are in the correct place and are in excellent condition.

$$\forall x(T(x) \rightarrow (P(x) \wedge D(x)))$$

Nothing is both in the correct place and in excellent condition.

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# Translations

Something is not in the cor

Everything is in the correct

All tools are in the correct |

A. $\forall x(\neg P(x) \wedge \neg D(x))$

B. $\neg\forall x(P(x) \wedge D(x))$

C. $\neg\exists x(P(x) \wedge D(x))$

D. $\exists x(\neg P(x) \wedge \neg D(x))$

E. None of the above.

Nothing is both in the correct place and in excellent condition.

There is not even one thing

One of the tools is not in the correct place, but it is in excellent condition.

# "De Morgan"-ish

*Rosen p. 45*

$$\neg \forall x P(x) \equiv \exists x\, (\neg P(x))$$

$$\neg \exists x P(x) \equiv \forall x\, (\neg P(x))$$

**Logical equivalence of quantified statements**: no matter which predicates are substituted and no matter which domains of discourse, the statements have the same truth value

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$\neg\exists x(P(x) \wedge D(x))$, or equivalently $\forall x(\neg P(x) \vee \neg D(x))$

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$$\exists x(\ T(x) \wedge \neg P(x) \wedge D(x)\ )$$

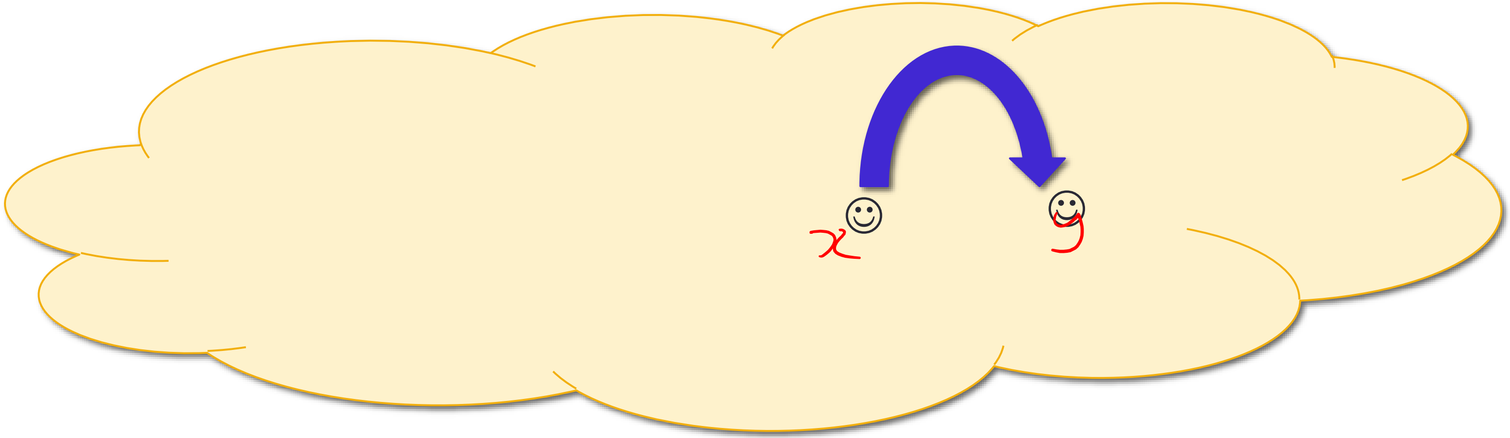
# Nested quantifiers

*Rosen p. 64 #3*

Q(x,y) "x has sent a text to y"  *domain: students in class*

(maybe x = y ?)

$$\exists x \exists y Q(x, y)$$



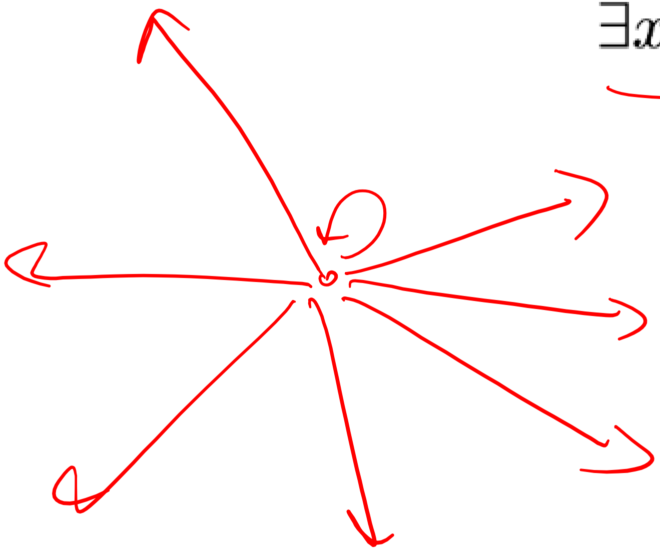
# Nested quantifiers

Rosen p. 64 #3

Q(x,y) "x has sent a text to y"

*domain: students in class*

$$\exists x \forall y Q(x,y)$$



Negation

$$\neg \exists x \forall y \, Q(x,y)$$

$$\equiv \forall x [\neg \forall y \, Q(x,y)]$$

$$\equiv \forall x \exists y \, \neg Q(x,y)$$

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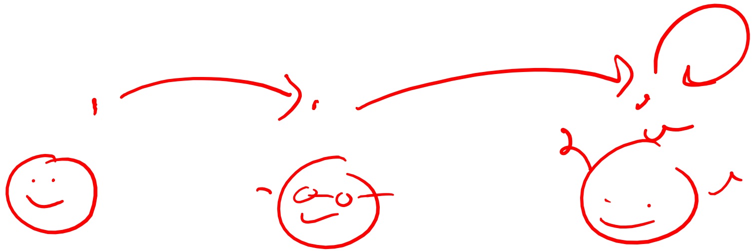

# Nested quantifiers

*Rosen p. 64 #3*

Q(x,y) "x has sent a text to y" *domain: students in class*

one student

$$\exists y \forall x Q(x,y)$$

(switch order)

# Nested quantifiers

*Rosen p. 64 #3*

Q(x,y) "x has sent a text to y" *domain: students in class*

$$\forall y \exists x Q(x, y)$$

# Nested quantifiers

*Rosen p. 64 #3*

Q(x,y) "x has sent a text to y" *domain: students in class*

$$\forall x \forall y Q(x, y)$$

# Evaluating quantified statements *Rosen p. 64 #1*

$$\forall x \exists y (x < y)$$

In which domain is this statement true?

A. All real numbers in the closed interval [0,1]. have max.
B. The set of integers {1,2,3}. have max
C. All real numbers. no max value
D. All positive real numbers.
E. All negative integers. max -1
F. All negative real numbers

Negation

$\neg \forall x \exists y (x < y) \equiv$

$\exists x \forall y (x \geq y)$

i.e. x is a maximum elt

Alt:

$\forall x (\exists y (x > y))$

# Evaluating quantified statements *Rosen p. 64 #1*

$$\forall x \forall y \exists z (xy = z)$$

In which domain is this statement **not** true?

A. All real numbers in the closed interval [0,1].
B. The set of integers {1,2,3}.
C. All real numbers.
D. All positive real numbers.
E. All positive integers.

# And in the other direction … *Rosen p. 66 #23*

- "The product of two negative real numbers is positive."

# And in the other direction … *Rosen p. 66 #23*

- "The difference between a real number and itself is zero."

Universe $\mathbb{R}$ $\qquad \forall x\,(x - x = 0)$

$$\forall x\,(E(x-x, 0))$$

$E(a, b)$ means $a = b$ $\qquad P(a) =$ "$a = 0$"

# And in the other direction … *Rosen p. 66 #23*

- "A negative real number does not have a square root that is a real number."

# And in the other direction … *Rosen p. 66 #23*

- "Every positive real number has exactly two square roots."

# Logical equivalence

*Rosen p. 646 #23*

Is it true that for every meaning of the predicate and every domain of discourse

$$\neg \exists x \forall y P(x, y) \quad \text{and} \quad \forall x \exists y \neg P(x, y)$$

have the same truth value?

# For next time…

- Get ready to prove!