CSE 20
DISCRETE MATH

New HW deadline: Saturday 11pm

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Prove propositional equivalences using truth tables
- Prove propositional equivalences using other known equivalences, e.g.
  - DeMorgan’s laws
  - Double negation laws
  - Distributive laws, etc.
- Compute the CNF and DNF of a given compound proposition.

(Some) Useful equivalences

Rosen p. 26-28

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \neg(p \vee q) \equiv \neg p \wedge \neg q$$ – De Morgan's

$$p \vee q \equiv q \vee p \qquad p \wedge q \equiv q \wedge p$$ – commutative

$$p \wedge F \equiv F \qquad p \vee T \equiv T$$

$$p \wedge T \equiv p \qquad p \vee F \equiv p$$

$$p \rightarrow q \equiv \neg p \vee q \qquad p \rightarrow q \equiv \neg q \rightarrow \neg p$$ – Contrapositive

…. 32 equivalences listed in book!

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) = p \wedge \neg q$

Can replace *p* and *q* with any (compound) proposition

(Some) Useful equivalences

Rosen p. 26-28

For constructing (minimal) circuits with specified gates

- **only NOTs?**
- **only ANDs?**

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

$p \wedge \neg p \equiv F$

…. 32 equivalences listed in book!

(Some) Useful equivalences

Rosen p. 26-28

For simplying and evaluating complicated compound propositions

- **Remove parentheses?**
- **Reduce subexpressions to simpler ones**

…. 32 equivalences listed in book!

(Some) Useful equivalences

Rosen p. 26-28

For devising proofs of statements

- **Translate using existing logical structure.**
- **Try to apply known proof strategy.**
- **Rewrite in equivalent way to apply additional proof strategies.**

(more on this later)

Sample equivalence proof

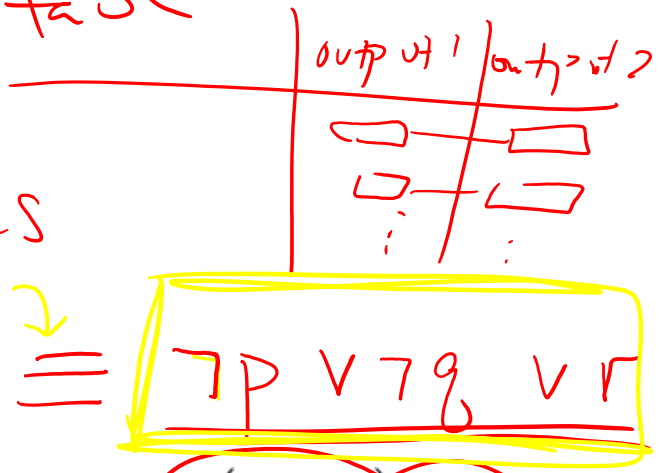
Prove that $(p \wedge q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$

Choice ① Write out truth table

Choice ② Apply known logical equivalences

Eqm in book

$(p \wedge q) \rightarrow r \equiv \neg(p \wedge q) \vee r \equiv$ DM $\neg p \vee \neg q \vee r$



Are these compound propositions logically equivalent to $(p \rightarrow q) \rightarrow r$? BONUS

No: Pf give example eg. P=F Q=? R=F

Other laws of equivalence

Rosen p. 29-31

Any compound proposition can be translated to one using …

A. only ANDs.
B. only ORs.
C. only IFs.
D. only NOTs.
E. None of the above

Other laws of equivalence

Rosen p. 35 #42-53

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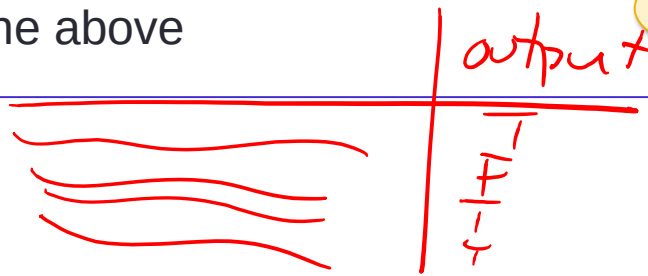
Functionally complete collection of connectives.



Functionally complete set of connectives *Rosen p. 35 #42-53*

Claim: The connectives AND, NOT are functionally complete.

Any compound proposition can be rewritten as a logically equivalent one that only has the operators AND, NOT

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Example: $p \wedge (q \wedge \neg r)$

Circuits?

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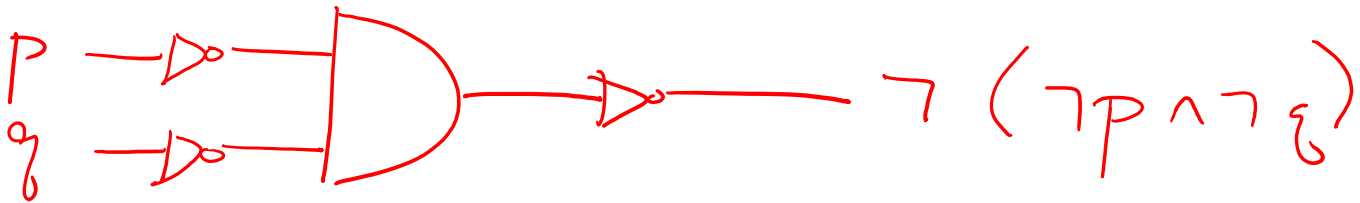
Example: $p \vee q$

$\neg(\neg(p \vee q)) \equiv (\neg p \wedge \neg q)$

Circuits?



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Claim: The connectives AND, NOT are functionally complete.

Any compound proposition can be rewritten as a logically equivalent one that only has the operators AND, NOT

Example: $p \rightarrow q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q)$

Circuit?

Functionally complete set of connectives *Rosen p. 35 #42-53*

Claim: The connectives AND, NOT are functionally complete.

Any compound proposition can be rewritten as a logically equivalent one that only has the operators AND, NOT

Example: $p \leftrightarrow q$

Circuit?

Functionally complete set of connectives *Rosen p. 35 #42-53*

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Any compound proposition can be rewritten as a logically equivalent one that only has the operators AND, NOT

Example: $p \oplus q$

Functionally complete set of connectives *Rosen p. 35 #42-53*

Rewriting compound propositions using only NOT, AND

1. Work from the inside out …
2. For each connective, replace it with an equivalent form that uses only NOT, AND:
   - If the connective is NOT or AND, do nothing.
   - If the connective is OR: replace $p \vee q$ with $\neg(\neg p \wedge \neg q)$
   - If the connective is IF..THEN: replace $p \rightarrow q$ with … $\neg(p \wedge \neg q)$
   - If the connective is IFF: replace $p \leftrightarrow q$ with … $\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)$
   - If the connective is XOR: replace $p \oplus q$ with … $\neg(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q))$

Functionally complete set of connectives *Rosen p. 35 #42-53*

Example: express $A \to (B \vee C)$ as a logically equivalent compound proposition that only uses ANDs and NOTs.

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$$A \to \neg(\neg B \wedge \neg C)$$

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Use $p \vee q \equiv \neg(\neg p \wedge \neg q)$ to rewrite intermediate step:

$$A \to \neg(\neg B \wedge \neg C)$$

Use $p \to q \equiv \neg(p \wedge \neg q)$ to rewrite:

$$\neg(A \wedge \neg(\neg(\neg B \wedge \neg C)))$$

Simplify double negation and use associativity:

$$\neg(A \wedge \neg B \wedge \neg C)$$

Going backwards

Given compound proposition, use

- Truth tables
- Logical equivalences

to compute truth values.

Reverse?

Given truth table settings, want compound proposition with that output.

E.g. Think back to HW2 Q2

CNF and DNF

Rosen p. 35 #42-53

Conjunctive normal form: AND of ORs (of variables or their negations).

Disjunctive normal form: OR of ANDs (of variables or their negations).

Which of the following is in CNF?

A. $p \vee q$ ∴ CNF, DNF

B. $\neg(p \vee q)$ NOT

C. $(\neg p \vee q) \wedge (p \vee \neg q)$

D. $(p \wedge q) \vee (\neg p \wedge \neg q)$ DNF

E. More than one of the above.

Edge case: A can be interpreted as an

- **AND (of itself), and as an**
- **OR (of itself)**

Reverse-engineering

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Reverse-engineering

Approach 1: classify rows based on one variable

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$$p \wedge (r \rightarrow q)$$

$$\neg p \wedge \neg q \wedge r$$

Reverse-engineering

Approach 2: algorithmically convert to normal form

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Reverse-engineering

Approach 2: algorithmically convert to normal form

DNF: when is output T?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

LAND IN THESE ROWS!

Reverse-engineering

Approach 2: algorithmically convert to normal form

DNF: when is output T?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
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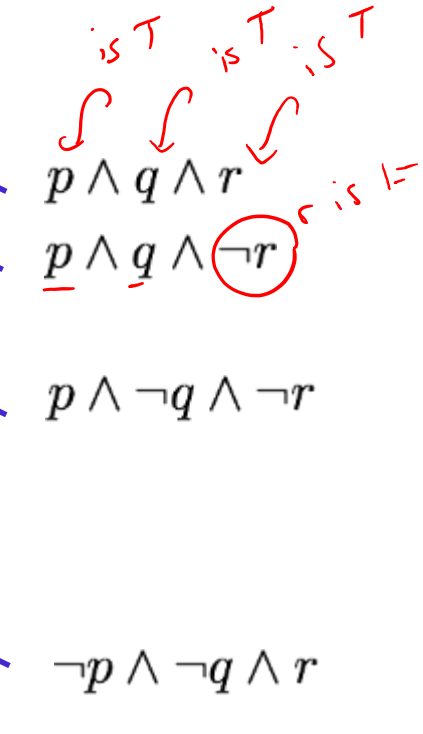

Reverse-engineering

Approach 2: algorithmically convert to normal form

DNF: when is output T?

p	*q*	*r*	*?*	
T	T	T	T	$p \wedge q \wedge r$
T	T	F	T	$p \wedge q \wedge \neg r$
T	F	T	F	
T	F	F	T	$p \wedge \neg q \wedge \neg r$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg p \wedge \neg q \wedge r$
F	F	F	F	

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$ DNF

Reverse-engineering

Approach 2: algorithmically convert to normal form

DNF: when is output T?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Reverse-engineering

Approach 2: algorithmically convert to normal form

CNF: when is output F?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

p not T ∨ q not F ∨ r not T

AVOID THESE ROWS!

Reverse-engineering

Approach 2: algorithmically convert to normal form

CNF: when is output F?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

AVOID THESE ROWS!

$$\neg(\neg p \wedge \neg q \wedge \neg r) \equiv p \vee q \vee r$$

Reverse-engineering

Approach 2: algorithmically convert to normal form

CNF: when is output F?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$\neg p \vee q \vee \neg r$

$p \vee \neg q \vee \neg r$

$p \vee \neg q \vee r$

$p \vee q \vee r$

Reverse-engineering

Approach 2: algorithmically convert to normal form

CNF: when is output F?

p	*q*	*r*	*?*
T	T	T	T
T	T	F	T
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T	F	F	T
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F	T	F	F
F	F	T	T
F	F	F	F

$\neg p \vee q \vee \neg r$

$p \vee \neg q \vee \neg r$

$p \vee \neg q \vee r$

$p \vee q \vee r$

CNF

$$(\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$$

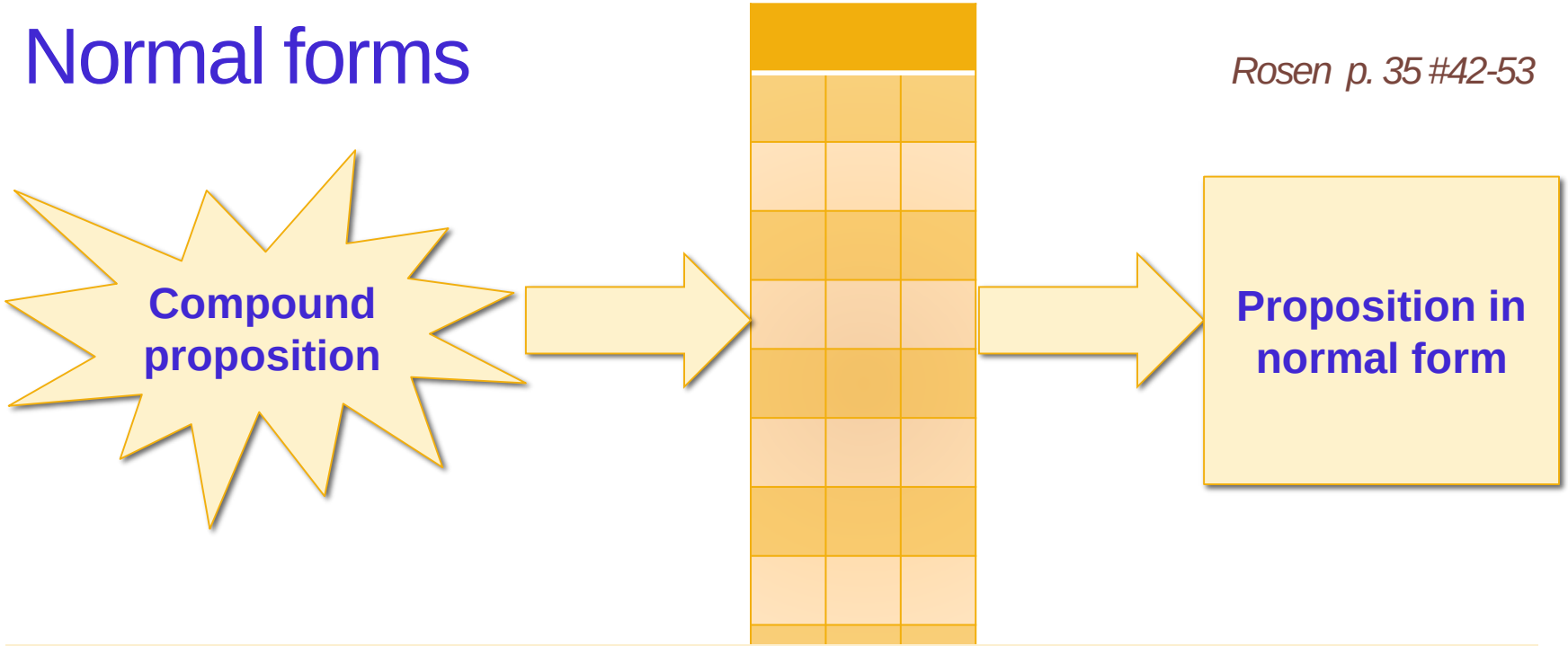
Payoff

- Any output column of a truth table (assignment of T/F to each combination of T/F input values) can be realized as a compound proposition.

- The collection $\vee \quad \wedge \quad \neg$ is functionally complete.

Normal forms

Rosen p. 35 #42-53



Added benefit: If want to reduce connectives further to prove a new collection of connectives is functionally complete, only need to consider those used in normal form.