CSE 20
DISCRETE MATH

There are 10 types of people in the world: those who understand binary and those who don't.

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Define the decimal, binary, hexadecimal, and octal expansions of a positive integer.
- Convert between expansions in different bases of a positive integer.
- Define and use the DIV and MOD operators.
- Describe and use algorithms for integer operations based on their expansions.

About you

CENTR 105: AB CENTR 115: BA

To change your remote frequency

1. Press and hold power button until flashing
2. Enter two-letter code
3. Checkmark / green light indicates success

How many people in this class have you met so far?

A. None.
B. Less than 5.
C. 5-10.
D. 10-15.
E. More than 15.

Algorithms!

From last time

An **algorithm** is a finite sequence of precise instructions for performing a computation or for solving a problem.

… arithmetic

… optimization

Representation

$$\log_2(32)$$

101

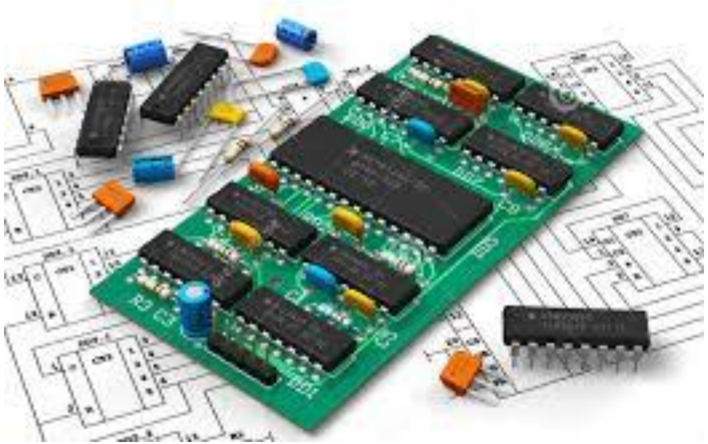
Five

5

Integer representations

Different contexts call for different representations.

Base 10



Base 2

Bases and algorithms

142 x **17**

Decimal expansion: sums of multiples of powers of 10

10^2 10^1 10^0

142 = 1 x 100 + 4 x 10 + 2 x 1

17 = 1 x 10 + 7 x 1

term

$241 \neq 142$

hundreds, tens, ones — hundreds, tens, ones

Powers of 10

Coefficient

Exponent

Base

$142 = 1 \times 10^2 + 4 \times 10^1 + 2 \times 10^0$

$17 = 1 \times 10^1 + 7 \times 10^0$

If n is a positive integer and we express its usual decimal expansion as sums of multiples of power of 10, then

A. The exponents of powers of 10 in the sum are all **positive**.
B. There is a **largest** exponent in the sum.
C. The **coefficient** of each power of 10 is nonnegative and less than 10.
D. More than one of the above.
E. All of the above.

Powers of b

Rosen Theorem 1 Sec 4.1 (p. 246)

For base b (integer greater than 1) and positive integer n there is a unique choice of

- k, a nonnegative integer
- $a_0, a_1, a_2, a_3, \ldots, a_{k-1}$ integers between 0 and b-1, where
- $a_{k-1} \neq 0$ and

$$n = a_{k-1}b^{k-1} + \ldots + a_1 b + a_0$$

Base expansion

Rosen Theorem 1 Sec 4.1 (p. 246)

Notation: for positive integer n

Write

$$n = (a_{k-1} \ldots a_1 \; a_0)_b$$

when

$$n = a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

Base b expansion of n

17

$(17)_{10} = (1\,0\,0\,0\,1)_2 \quad (k=5)$

eights ones

$= (2\,1)_8 \quad (k=2)$

sixteens ones

$= (1\,1)_{16} \quad (k=2)$

coeff : $0 \rightarrow b-1$

Hexadecimal coefficients

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7

- 8
- 9
- A value 10
- B value 11
- C 12
- D 13
- E 14
- F 15

Base expansion

In what base **could** this expansion be

$$(1401)_?$$

A. Binary (base 2)
B. Octal (base 8)
C. Decimal (base 10)
D. Hexadecimal (base 16)
E. More than one of the above

Base expansion

In what base **could** this expansion be

$(1401)_?$

A. Binary (base 2)
B. Octal (base 8) — Value: $1*8^3+4*8^2+1=768$
C. Decimal (base 10) — Value: $1*10^3+4*10^2+1=1401$
D. Hexadecimal (base 16) — Value: $1*16^3+4*16^2+1=5121$
E. More than one of the above

Arithmetic

Addition algorithm in Rosen 4.2 (p. 250)

$$
\begin{array}{r}
(1401)_{16} \\
+\ (999)_{16} \\
\hline
(1D9A)_{16}
\end{array}
$$

$$
\begin{array}{r}
(1401)_{16} \\
\times\ (64)_{16} \\
\hline
5004 \\
+\ 060 \\
\hline
\end{array}
$$

Algorithm: constructing base b expansion

pos ints   length of rep'n

Input n,b   **Output** k, coefficients in expansion

int > 1

- English description.

- Pseudocode.

Algorithm: constructing base b expansion

Input n,b **Output** k, coefficients in expansion

- English description.

  Find k

  Work down to find a_{k-1}, then a_{k-2}, etc.

Algorithm: constructing base b expansion

Input n,b   **Output** k, coefficients in expansion

- English description.
  - Find k
  - Work down to find a_{k-1}, then a_{k-2}, etc.

What's the length of the base b expansion of n?
rephrase as:
What's the smallest power of b that is bigger than n?

Algorithm: constructing base b expansion

Input n,b **Output** k, coefficients in expansion

English description:

Find k by computing successive powers of b until find smallest k such that

$$b^{k-1} \leq n < b^k$$

Initialize value remaining v := n

For each value of i from 1 to k

Set a_{k-i} to be the largest number between

0 and b-1 for which $a_{k-i}\, b^{k-i} \leq v$.

Update v := v - $a_{k-i}\, b^{k-i}$

$2\cdot 9 + 1\cdot 3 + 0 =$

$1\cdot 9 + 1\cdot 3 + 1 =$

What's the the base 3 expansion of 17?

A. $(10001)_3$
B. $(210)_3$
C. $(111)_3$
D. $(222)_3$
E. None of the above.

Algorithm: constructing base b expansion

Input n,b **Output** k, coefficients in expansion

English description:

Find k by computing successive powers of b until find smallest k such that

$$b^{k-1} \leq n < b^k$$

Initialize value remaining v := n

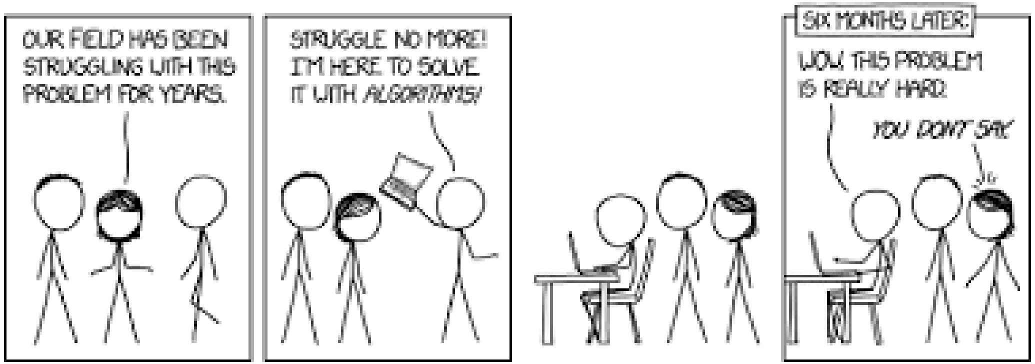
For each value of i from 1 to k

Set a_{k-i} to be the largest number between

0 and b-1 for which $a_{k-i}\, b^{k-i} \leq v$.

Update v := v - $a_{k-i}\, b^{k-i}$

Definite? Finite? Correct? ***Challenge: translate to pseudocode!***


OUR FIELD HAS BEEN STRUGGLING WITH THIS PROBLEM FOR YEARS.
STRUGGLE NO MORE! I'M HERE TO SOLVE IT WITH ALGORITHMS!
SIX MONTHS LATER:
WOW, THIS PROBLEM IS REALLY HARD.
YOU DON'T SAY.

xkcd

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

Idea: Find smallest digit first, then next smallest, etc.

…. **but how?**

Reminder: Divisibility

Rosen p. 237-239

Theorem: For n an integer and d a positive integer, there are unique integers q and r with $0 \le r < d$ and $n = qd + r$.

Notation: q = n **div** d   r = n **mod** d

Quotient when divide n by b

Remainder when divide n by b

What is 24 **div** 5? What is 15 **mod** 5?

A. 4, 0
B. 3, 0
C. 4, 3
D. 3, 3
E. I don't know.

Bases and Divisibility

Rosen p. 237-239

When k>1

$$n = a_{k-1}b^{k-1} + \ \ldots + a_1 b + a_0$$

Bases and Divisibility

Rosen p. 237-239

When k>1

$$n = a_{k-1}b^{k-1} + \ \ldots + a_1 b + a_0$$

$$= b\,(a_{k-1}b^{k-2} + \ \ldots + a_1) + a_0$$

d

q = n div d

r = n mod d

coeffs a_i in [0, b-1]

Bases and Divisibility

Rosen p. 237-239

Useful fact:

if $n = (a_{k-1}\ldots a_0)_b$ then $bn = (a_{k-1}\ldots a_0 0)_b$

Why?

How is this useful?

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

Idea: Use n **mod** b to compute least significant digit.

Use n **div** b to compute new integer whose expansion we need. Repeat.

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

Pseudocode:

procedure *base b expansion*$(n, b:$ pos ints with $b > 1)$

1. $q := n$
2. $k := 0$
3. **while** $q \neq 0$
4. $\quad a_k := q \textbf{ mod } b$
5. $\quad q := q \textbf{ div } b$
6. $\quad k := k + 1$
7. **return** $(a_{k-1}, \ldots, a_1, a_0)$

Algorithm: constructing base b expansion *Rosen p. 249*

procedure *base b expansion*$(n, b:$ pos ints with $b > 1)$

1. $q := n$
2. $k := 0$
3. **while** $q \neq 0$
4. $\quad a_k := q$ **mod** b
5. $\quad q := q$ **div** b
6. $\quad k := k + 1$
7. **return** $(a_{k-1}, \ldots, a_1, a_0)$

n	*b*	*q*	*k*	a_k
17	**3**	17	0	17 **mod** 3 = 2
		17 **div** 3 = 5	1	5 **mod** 3 = 2
		5 **div** 3 = 1	2	1 **mod** 3 = 1
		1 **div** 3 = 0	3	return!

Definite? Finite? Correct?

Terminology

- Base b expansion:
  - Coefficients between 0 and b-1 (inclusive)
  - Leading coefficient nonzero

- **Fixed-width** base b expansion:
  - Coefficients between 0 and b-1 (inclusive)
  - Pad leading coefficients as 0 to match desired width

Representing more?

- Base b expansions can express any **positive integers**
- Fixed width base b expasions can express **nonnegative integers** [0, b^k-1]

- What about
  - negative integers?
  - rational numbers?
  - other real numbers? *stay tuned for CSE 30, CSE 140*

Reminders

- Homework 1 due Friday at 11pm
  - Pseudocode and algorithms + number representations
- Office hours available
  - Group
  - One-on-one

There are 10 types of people in the world: those who understand ternary, those who don't, and those who mistake it for binary

(Ternary means base 3)