Tomorrow: Discussion +
Review Quiz due
Saturday: HW7 due

# CSE 20
# DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

# Today's learning goals

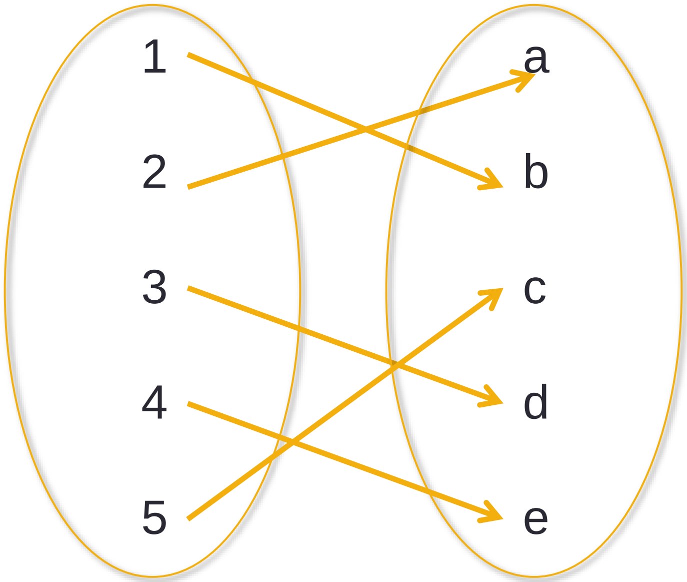
- Define and compute the cardinality of a set.
- Use functions to compare the sizes of sets.
- Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
- Explain the central idea in Cantor's diagonalization argument.

# Cardinality

*Rosen Section 2.5*

For all sets, we **define**

$|A| = |B|$ if and only if there is a bijection between them.



# Cardinality

Countable Sets

Rosen Defn 3 p. 171

- Finite sets $\emptyset$ $\{1\}$ $\{1,2\}$ $\mathcal{P}(\{1,2,3\})$ — $|A| = n$ for some nonnegative int $n$
- Countably infinite sets — $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)
- ~~Uncountable sets~~ — Infinite but not in bijection with $\mathbf{Z}^+$

The set of all prime #s $\subseteq \mathbb{Z}^+$ so ctbly infinite

$$|\mathcal{P}(\{1,2,3\})| = 2^{|\{1,2,3\}|} = 2^3 = 8 \in \mathbb{N}$$

# Lemmas … how would you prove each one?

- If A and B are countable sets, then A U B is countable.

  *Theorem 1, p. 174*

- If A and B are countable sets, then A x B is countable.
- If A is finite, then A* is countable.
- If A is a subset of B, to show that |A| = |B|, it's enough to give a 1-1 function from B to A or an onto function from A to B.

  *Exercise 22, p. 176*

  e g $Z^+$

- If A is a subset of a countable set, then it's countable.

  *Exercise 16, p. 176*

- If A is a superset of an uncountable set, then it's uncountable.

  *Exercise 15, p. 176*

# Cardinality

*Rosen p. 172*

- Countable sets    A is finite or $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

*Examples:*  $\emptyset$  $\{x \in \mathbb{Z} | x^2 = 1\}$  $\mathcal{P}(\{1,2,3\})$  $\mathbb{Z}^+$
*and also …*

- the set of **odd positive** integers — Example 1
- the set of **all integers** — Example 3
- the set of **positive rationals** — Example 4
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** {0,1}*

# Cardinality

*Rosen p. 172*

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- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** $\{0,1\}^*$

Proof strategies?

- List out all and only set elements (with or without duplication)
- Give a one-to-one function from A to (a subset of) a set known to be countable

# There is an uncountable set! *Rosen example 5, page 173-174*

## Cantor's diagonalization argument

**Theorem: For every set A,** $|A| \neq |\mathcal{P}(A)|$

# There is an uncountable set! *Rosen example 5, page 173-174*

## Cantor's diagonalization argument

**Theorem: For every set A,** $|A| \neq |\mathcal{P}(A)|$

$\neg\exists$ bij from A to $\mathcal{P}(A)$

**An example to see what is necessary. Consider A = {a,b,c}.**

**What would we need to prove that |A| = |P(A)|?**

a
b
c

$\emptyset$
$\{a\}$
$\{b\}$ . . .

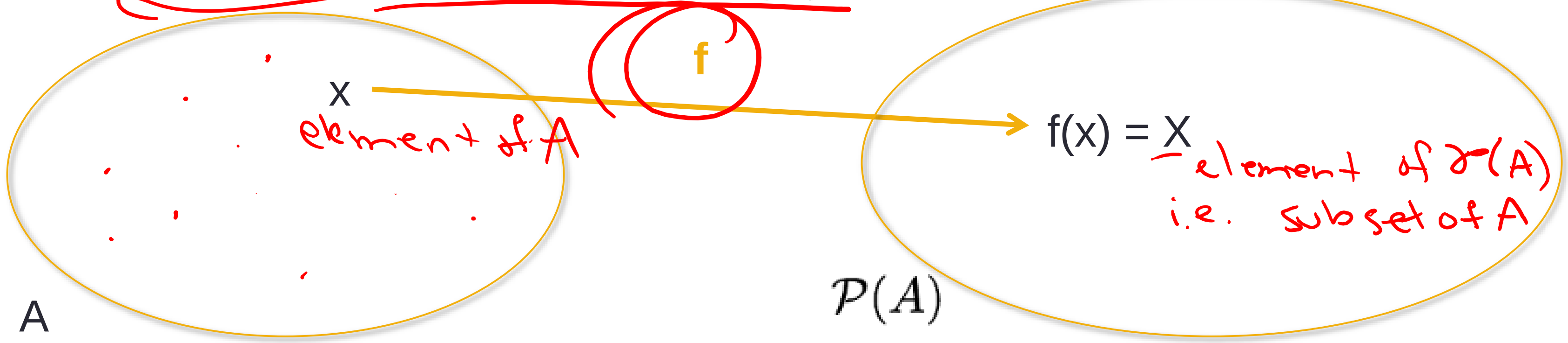
# There is an uncountable set! *Rosen example 5, page 173-174*

## Cantor's diagonalization argument

**Theorem: For every set A,** $|A| \neq |\mathcal{P}(A)|$

**Proof:** (Proof by contradiction)

Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \rightarrow \mathcal{P}(A)$.



# There is an uncountable set! *Rosen example 5, page 173-174*

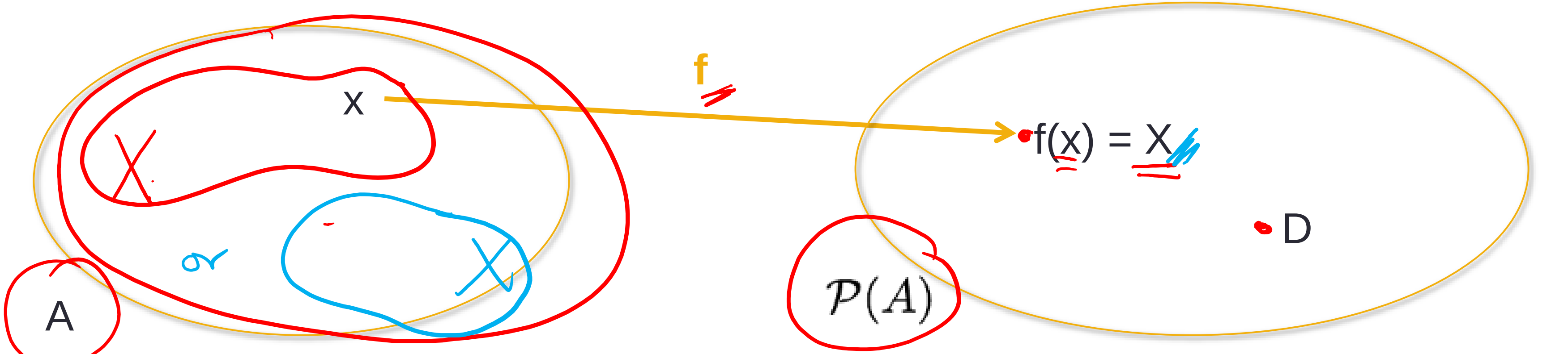
## Cantor's diagonalization argument

Consider the subset $\boxed{D}$ of A defined by, for each a in A:

Def of D: $$a \in D \quad \text{iff} \quad a \notin f(a)$$

Case 1 : $x \notin D$

Case 2 : $x \in D$



# There is an uncountable set!

*Rosen example 5, page 173-174*

A $\mathcal{P}(A)$

$d \in A$ f D set

## Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A: defining pred for D

$$F \quad \boxed{\begin{array}{c} a \in D \\ d \in D \end{array}} \quad \text{iff} \quad \begin{array}{c} a \notin f(a) \\ d \notin f(d) \end{array} \quad F$$

Define d to be the pre-image of D in A under f $\quad f(d) = D$

**Is d in D?**

$d \in D \land d \notin D$

- Case ① If yes, then by definition of D, $d \notin f(d) = D$ **a contradiction!**
- Case ② Else, by definition of D, $\neg(d \notin f(d))$ so $d \in f(d) = D$ **a contradiction!**

$d \notin D \land d \in D$

# Cardinality

*Rosen p. 172*

- Uncountable sets

Infinite but not in bijection with $\mathbf{Z}^+$

*Examples:* the power set of any countably infinite set

*and also …*

- the set of **real** numbers — Example 5
- (0,1) — Example 6 (++)
- (0,1] — Example 6 (++)

[0,1]

*Exercises 33, 34*

# Why the real numbers?

(0,1)

If this little interval is already uncountable,
then R is definitely uncountable!

$17 = 2^4 + 2^0 = (10001)_2$

$0.5 = 0.10000\ \ldots$ maps to $\{1\}$

$0.1 = 0.00011\ \ldots$ maps to $\{4, 5, \ldots\}$

$0.\, b_1\, b_2\, b_3\, b_4\ \ldots.$

binary expansion of number

A. $\{4\}$

B. $\{5\}$

C. $\{1,2,3,4,5\}$

D. $\{4,5\}$

E. ???

# Why the real numbers?

(0,1)

If this little interval is already uncountable,
then R is definitely uncountable!

*"Looks like" a power set of a countably infinite set?*

ignore

$\boxed{0.}\ b_1\ b_2\ b_3\ b_4\ \ldots.$ binary expansion of number

maps to flag about $1 \in$ set

$\{ x \mid x \text{ is a positive integer and } b_x \text{ is } 1\}$

**Conclude**: | (0,1) | = | power set of $\mathbf{Z}^+$ |

# Diagonalization

*Example 5 Rosen p. 173*

**Theorem: The set (0,1) is uncountable**

**Proof:** (Proof by contradiction) Assume towards a contradiction that (0,1) is countable. By definition, that means there is a **bijection** which lists all real numbers in this interval.

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*Example 5 Rosen p. 173*

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$f(1) = r_1 = 0.\ b_{11}\ b_{12}\ b_{13}\ b_{14}\ \ldots$

$f(2) = r_2 = 0.\ b_{21}\ b_{22}\ b_{23}\ b_{24}\ \ldots$

$f(3) = r_3 = 0.\ b_{31}\ b_{32}\ b_{33}\ b_{34}\ \ldots$

$f(4) = r_4 = 0.\ b_{41}\ b_{42}\ b_{43}\ b_{44}\ \ldots$

We're going to find a number **d** that is not in this list.

# Diagonalization

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$0.0$ ______

$0.1$ ______

$\frac{1}{\pi}$ $f(1) = r_1 = 0.\, b_{11}\ b_{12}\ b_{13}\ b_{14}\ \ldots$

$0.1$ $f(2) = r_2 = 0.\, b_{21}\ b_{22}\ b_{23}\ b_{24}\ \ldots$

$\frac{1}{2}$ $f(3) = r_3 = 0.\, b_{31}\ b_{32}\ b_{33}\ b_{34}\ \ldots$

$\frac{1}{\sqrt{2}}$ $f(4) = r_4 = 0.\, b_{41}\ b_{42}\ b_{43}\ b_{44}\ \ldots$

We're going to find a number **d** that is not in this list.

$$d = 0.\, b_1\ b_2\ b_3\ b_4\ \ldots$$

where $b_i = 1 - b_{ii}$ . By this definition: **d can't equal any f(i).** So: **f is not onto!**

# What about the irrational numbers?

**Claim**: The set of irrational numbers **is / isn't** countable.

**Proof:**