Reminders:
Discussion & Review Quiz tomorrow
HW7 due Saturday
Today's learning goals

• Define and compute the cardinality of a set.
• Use functions to compare the sizes of sets.
• Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
• Explain the central idea in Cantor’s diagonalization argument.
For all sets, we define
\[ |A| = |B| \text{ if and only if there is a bijection between them.} \]
Cardinality

- Finite sets
  \[ |A| = n \text{ for some nonnegative int } n \]
- Countably infinite sets
  \[ |A| = |\mathbb{Z}^+| \] (informally, can be listed out)
- Uncountable sets
  Infinite but not in bijection with \( \mathbb{Z}^+ \)

Set of all prime numbers \( \leq \mathbb{Z}^+ \) and infinite

\[ \mathcal{O}(\{1,2,3\}) = 2^{\#(\{1,2,3\})} = 2^3 = 8 \in \mathbb{N} \]
Lemmas ... how would you prove each one?

- If $A$ and $B$ are countable sets, then $A \cup B$ is countable. 
  \textit{Theorem 1, p. 174}

- If $A$ and $B$ are countable sets, then $A \times B$ is countable.

- If $A$ is finite, then $A^*$ is countable.

- If $A$ is a subset of $B$, to show that $|A| = |B|$, it's enough to give a $1-1$ function from $B$ to $A$ or an onto function from $A$ to $B$. 
  \textit{Exercise 22, p. 176}

- If $A$ is a subset of a countable set, then it's countable. 
  \textit{Exercise 16, p. 176}

- If $A$ is a superset of an uncountable set, then it's uncountable. 
  \textit{Exercise 15, p. 176}
Cardinality

- **Countable sets**  \( A \) is finite or \( |A| = |\mathbb{Z}^*| \) (informally, can be listed out)

**Examples:**

- the set of **odd positive** integers
- the set of **all integers**
- the set of **positive rationals**
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** \( \{0,1\}^* \)
Cardinality

- Countable sets: \( A \) is finite or \( |A| = |\mathbb{Z}^*| \) (informally, can be listed out)

Examples:
- the set of odd positive integers
- the set of all integers
- the set of positive rationals
- the set of negative rationals
- the set of rationals
- the set of nonnegative integers
- the set of all bit strings \( \{0,1\}^* \)

Proof strategies?
- List out all and only set elements (with or without duplication)
- Give a one-to-one function from \( A \) to (a subset of) a set known to be countable
There is an uncountable set!  

Rosen example 5, page 173-174

Cantor's diagonalization argument

Theorem: For every set $A$, $|A| \neq |\mathcal{P}(A)|$
There is an uncountable set!  

Rosen example 5, page 173-174

Cantor's diagonalization argument

Theorem: For every set $A$, $|A| 
eq |\mathcal{P}(A)|$

An example to see what is necessary. Consider $A = \{a,b,c\}$. What would we need to prove that $|A| = |\mathcal{P}(A)|$?
There is an uncountable set!  

Rosen example 5, page 173-174

Cantor's diagonalization argument

Theorem: For every set $A$, $|A| \neq |\mathcal{P}(A)|$

Proof: (Proof by contradiction)
Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a bijection $f: A \rightarrow \mathcal{P}(A)$.

\[ f(x) = \mathcal{P}(A) \ni x \leq A \]

\[ f(x) = X \]
There is an uncountable set!  

Rosen example 5, page 173-174

Cantor's diagonalization argument

Consider the subset $D$ of $A$ defined by, for each $a$ in $A$:

$$a \in D \quad \text{iff} \quad a \notin f(a)$$

$$D = \{ a \in A \mid a \notin f(a) \}$$
Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

\[ a \in D \quad \text{iff} \quad a \notin f(a) \]

Define d to be the pre-image of D in A under f.

Is d in D?

- If yes, then by definition of D, \( d \notin f(d) = D \)
- Else, by definition of D, \( \neg(d \notin f(d)) \) so \( d \in f(d) = D \)

There is an uncountable set! Rosen example 5, page 173-174
Cardinality

• Uncountable sets

Examples: the power set of any countably infinite set

- the set of real numbers
- (0,1)
- (0,1]
- \([0,1]\)

Infinite but not in bijection with \(\mathbb{Z}^+\)

Examples:

Example 5
Example 6 (++)
Example 6 (++)

Exercises 33, 34
Why the real numbers?

If this little interval is already uncountable, then R is definitely uncountable!

\[ l = \frac{1}{2^4} + \frac{1}{2^5} = \frac{10001}{2^5} \]

0.5 = 0.10000 ... maps to \{ \frac{3}{4} \}

0.1 = 0.00011 ... maps to \{ 4, 5, \ldots \}

0. b_1 b_2 b_3 b_4 ....

generalization

binary expansion of number
Why the real numbers?

(0,1) ... If this little interval is already uncountable, then R is definitely uncountable!

"Looks like" a power set of a countably infinite set?

0. $b_1 \ b_2 \ b_3 \ b_4 \ \ldots.$ binary expansion of number $b_1 w_0 . 1$

maps to

$\{ \ x \ | \ x \ is \ a \ positive \ integer \ and \ b_x \ is \ 1\}$

set of pos.nts set of pos.nts positions

Conclude: $| (0,1) | = | power \ set \ of \ Z^+ |$
Diagonalization

Theorem: The set (0,1) is uncountable

Proof: (Proof by contradiction) Assume towards a contradiction that (0,1) is countable. By definition, that means there is a bijection which lists all real numbers in this interval.
Theorem: The set (0,1) is uncountable

Proof: (Proof by contradiction) Assume towards a contradiction that (0,1) is countable. By definition, that means there is a bijection which lists all real numbers in this interval.

\[
\begin{align*}
  f(1) &= r_1 = 0.\, b_{11} b_{12} b_{13} b_{14} \ldots \\
  f(2) &= r_2 = 0.\, b_{21} b_{22} b_{23} b_{24} \ldots \\
  f(3) &= r_3 = 0.\, b_{31} b_{32} b_{33} b_{34} \ldots \\
  f(4) &= r_4 = 0.\, b_{41} b_{42} b_{43} b_{44} \ldots \\
\end{align*}
\]

We're going to find a number \( d \) that is not in this list.
Diagonalization

Theorem: The set (0,1) is uncountable

Proof: (Proof by contradiction) Assume towards a contradiction that (0,1) is countable. By definition, that means there is a bijection which lists all real numbers in this interval.

\[ f(1) = r_1 = 0. \overline{b_{11}} b_{12} b_{13} b_{14} \ldots \]
\[ f(2) = r_2 = 0. \overline{b_{21}} b_{22} b_{23} b_{24} \ldots \]
\[ f(3) = r_3 = 0. \overline{b_{31}} b_{32} b_{33} b_{34} \ldots \]
\[ f(4) = r_4 = 0. \overline{b_{41}} b_{42} b_{43} b_{44} \ldots \]

We're going to find a number \( d \) that is not in this list.

\[ d = 0. b_1 b_2 b_3 b_4 \ldots \]

where \( b_i = 1 - b_{ii} \). By this definition: \( d \) can't equal any \( f(i) \). So: \( f \) is not onto!
What about the irrational numbers?

Claim: The set of irrational numbers is / isn't countable.

Proof: