CSE 20
DISCRETE MATH

Reminder: Discussion sections are being held this week.

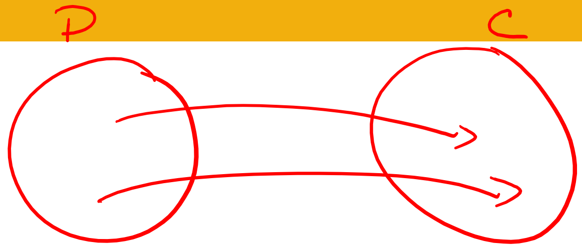
Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Define and compute the cardinality of a set.
- Use functions to compare the sizes of sets.
- Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
- Explain the central idea in Cantor's diagonalization argument.

Functions



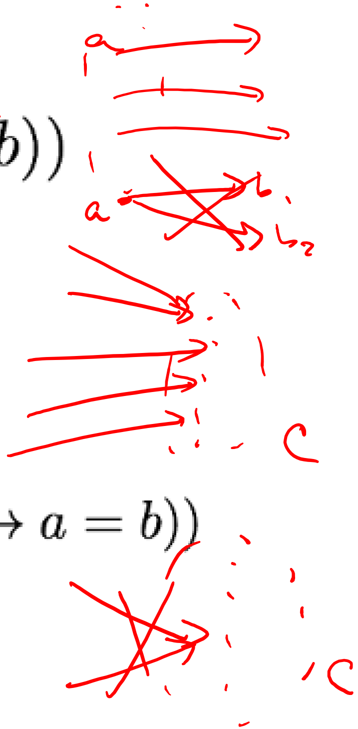
Rosen Sec 2.3; p. 138

Function f: D → C means domain D, codomain C, plus rule

Well-defined $\forall a(\overbrace{a \in D}^{\text{HYP}} \rightarrow \overbrace{\exists! b\,(b \in C \wedge f(a) = b)}^{\text{CONC}})$

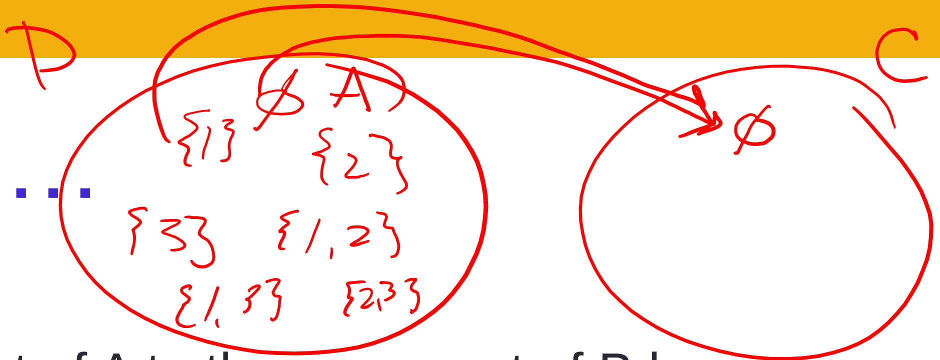
Onto $\forall b(b \in C \rightarrow \exists a\,(a \in D \wedge f(a) = b))$

One-to-one $\forall a \forall b((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$



(To prove F, only need counterex)

Proving a function is …



Let A = {1,2,3} and B = {2,4,6}.

Define a function from the power set of A to the power set of B by:

$$f : \mathcal{P}(A) \to \mathcal{P}(B)$$
$$f(X) = X \cap B$$

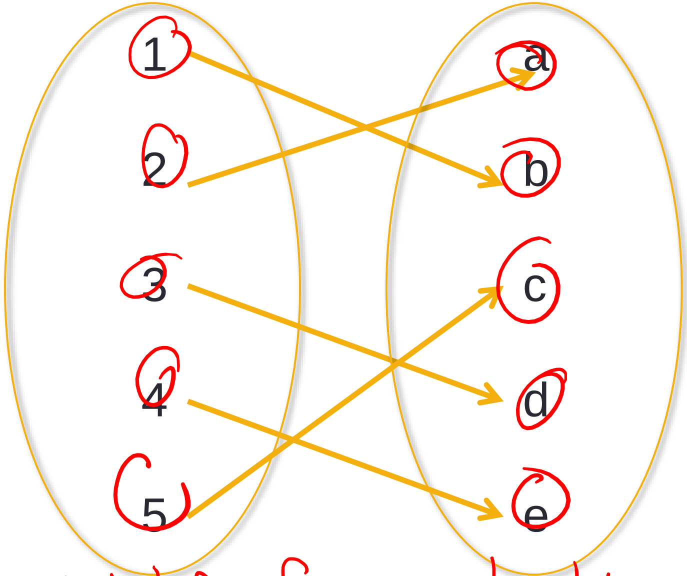
$f(\emptyset) = \emptyset \cap B$
$= \emptyset$

$f(\{1\}) = \{1\} \cap \{2,4,6\}$
$= \emptyset$

- **Well-defined?** – D has 8 elts, check each one
  or consider $X \in D$ i.e. $X \subseteq A$
  apply def $f(X) = X \cap B$
  WTS $f(X) \in C = \mathcal{P}(B)$
  i.e. $f(X) \subseteq B$ True by def of $\cap$
- **Onto?** N
- **One-to-one?** N
  counterex (above)

One-to-one + onto

Rosen p. 144



Fact: If f, g both satisfy then both are well-defined bijections

one-to-one correspondence

bijection

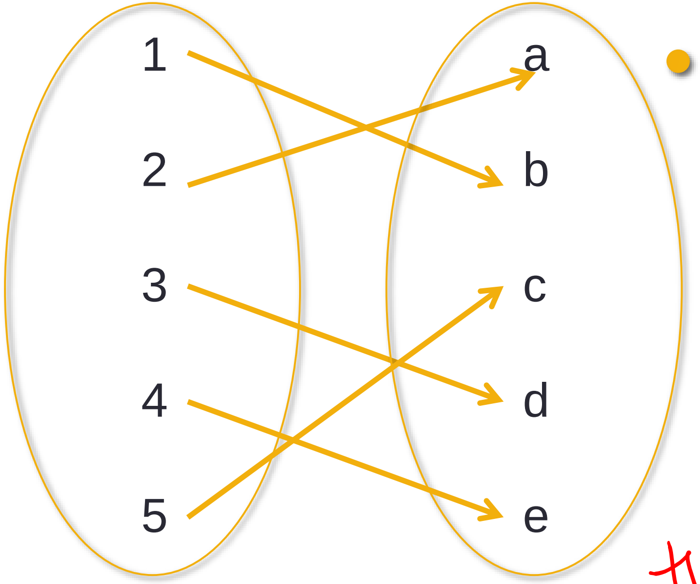
invertible

The **inverse** of a function f: A→B is the function g: B→A such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144



Fact: for finite sets A and B, there is a bijection between them if and only if $|A| = |B|$.

Cool fact: if have finite sets A, B and $|A| = |B|$ then for any well defined function $f: A \to B$ f is 1-1 iff it's onto

Finite $\{1,2,3\} \subsetneq \{1,2,3,4\}$

Beyond finite sets

Rosen Section 2.5

For all sets, we **define**

$|A| = |B|$ if and only if there is a bijection between them.

add one element

$\mathbb{N}$ 0, 1, 2, ...

$\mathbb{Z}^+$ 1, 2, 3, . . .

$f: \mathbb{N} \to \mathbb{Z}^+$ $f(x) = x+1$ bij

$\mathbb{Z}$ . . . −2, −1, 0, 1, 2, ...

add all negs

0, 1, −1, 2, −2, 3, −3, . . .

$f: \mathbb{N} \to \mathbb{Z}$ $f(x) = \{$ if x odd / if x even

Which of the following is true?

A. $|\mathbb{Z}| = |\mathbb{N}|$

B. $|\mathbb{N}| = |\mathbb{Z}^+|$

C. $|\mathbb{Z}| = |\{0,1\}^*|$

D. All of the above.

E. None of the above.

Cardinality

Rosen Defn 3 p. 171

- Finite sets $|A| = n$ for some nonnegative int n
- Countably infinite sets $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

"Smallest" infinite set

Sizes and subsets

*Rosen Theorem 2, p 174 *More on HW**

For all sets A, B we say

$|A| \leq |B|$ if there is a one-to-one function from A to B.

$|A| \geq |B|$ if there is an onto function from A to B.

Cantor-Schroder-Bernstein Theorem: $|A| = |B|$ iff $|A| \leq |B|$ and $|A| \geq |B|$

Which of the following is true?

A. If A is a subset of B then $|A| \leq |B|$ Pf in HW

B. If A is a proper subset of B then $|A| \neq |B|$

C. If A is a subset of B then $|A| = |B|$

D. None of the above.

E. I don't know

Countere ex: $A = B$

Counter ex $A = \mathbb{Z}^+$, $B = \mathbb{N}$

Counterex $A = \{1,2,3\}$ $B = \{1,2,3,4\}$

$\mathbb{Q}^+ = \left\{ \frac{p}{q} \;\middle|\; \begin{array}{l} p \in \mathbb{N} \\ q \in \mathbb{Z}^+ \end{array} \right\}$

$\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{0}{2}, \ldots$

Beyond finite sets

Rosen Section 2.5

Duplicates!

$f: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ onto but not 1-1

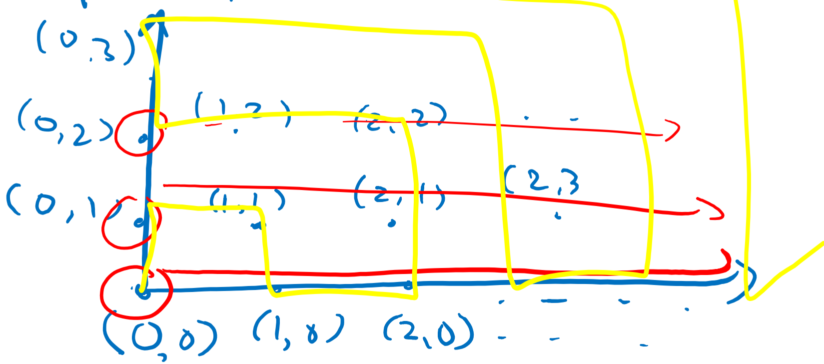
$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}| \geq |\mathbb{Q}^+| \geq |\mathbb{N}|$

For all sets, we say

|A| = |B| if and only if there is a bijection between them.

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$



$(0,0), (0,1), (1,1), (1,0), \ldots$

Which of the following is true?

A. $|\mathbb{Q}| = |\mathbb{Q}^+|$
B. $|\mathbb{Q}^+| = |\mathbb{N} \times \mathbb{N}|$
C. $|\mathbb{N}| = |\mathbb{Q}|$
D. All of the above.
E. None of the above.

Cardinality

Rosen Defn 3 p. 171

- Finite sets $|A| = n$ for some nonnegative int n
- Countably infinite sets $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)
- Uncountable sets Infinite but not in bijection with $\mathbf{Z}^+$

Looking ahead:
$\mathbb{R}$ is uncountable
so $\mathbb{R} - \mathbb{Q}$ is uncountable

Cardinality

Rosen Defn 3 p. 171

- Finite sets

|A| = n for some nonnegative int n

Which of the following sets is **not** finite?

A. $\emptyset$

B. $[0,1]$

C. $\{x \in \mathbb{Z} | x^2 = 1\}$

D. $\mathcal{P}(\{1,2,3\})$

E. None of the above (they're all finite)

Cardinality

Rosen p. 172

- Countable sets A is finite or $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$ $\mathcal{P}(\{1,2,3\})$ $\mathbb{Z}^+$

and also …

- the set of **odd positive** integers Example 1
- the set of **all integers** Example 3
- the set of **positive rationals** Example 4
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** {0,1}*

Cardinality

Rosen p. 172

- Countable sets   A is finite or $|A| = |\mathbb{Z}^+|$ (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$
and also …
- the set of **odd positive** integers
- the set of **all integers**
- the set of **positive rationals**
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** {0,1}*

Proof strategies?

- List out all and only set elements (with or without duplication)
  ok
- Give a one-to-one function from A to (a subset of) a set known to be countable

Proving countability

Pf in 2.5

Which of the following is **not** true?

A. If A and B are both countable then $A \cup B$ is countable.

B. If A and B are both countable then $A \cap B$ is countable.

C. If A and B are both countable then $A \times B$ is countable.

D. If A is countable then P(A) is countable.

E. None of the above

Grid
Spiral

There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

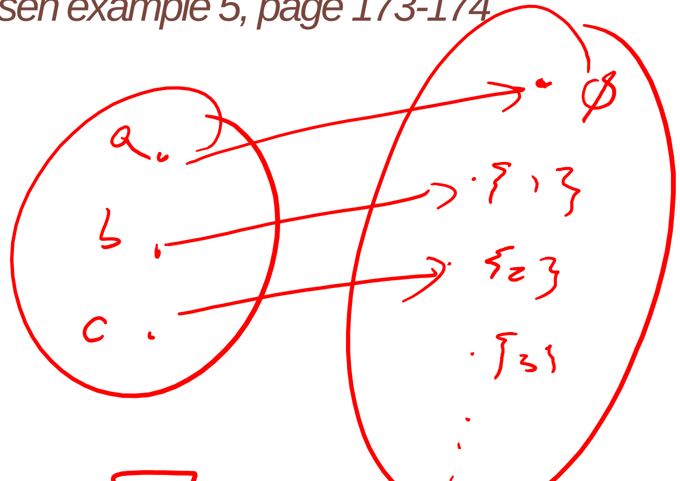
<u>Cor</u> $\mathcal{P}(\mathbb{Z}^+)$ is uncountable

There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

$\emptyset$ $\qquad \mathcal{P}(\emptyset) = \{\emptyset\}$



An example to see what is necessary. Consider A = {a,b,c}.

What would we need to prove that |A| = |P(A)|?

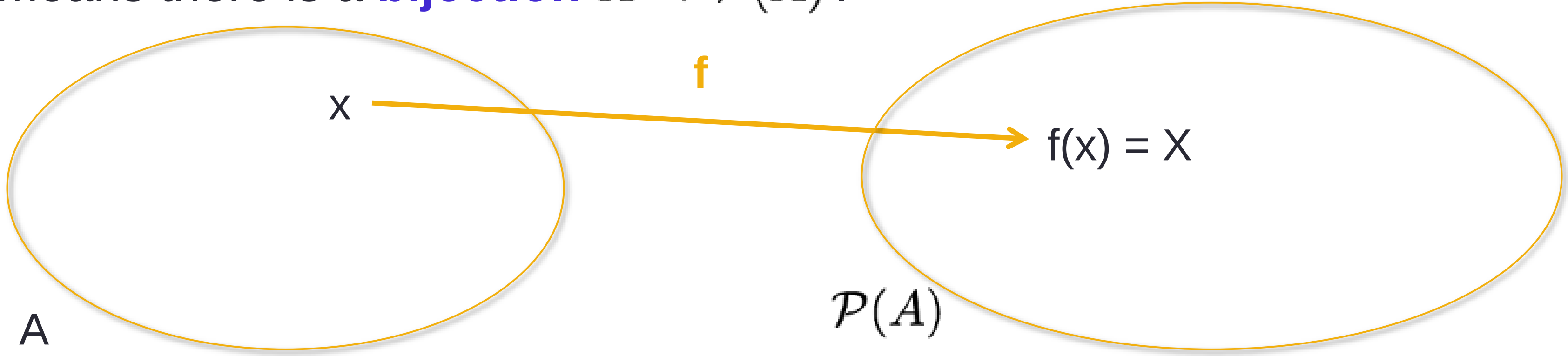
There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

Proof: (Proof by contradiction)

Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \to \mathcal{P}(A)$.

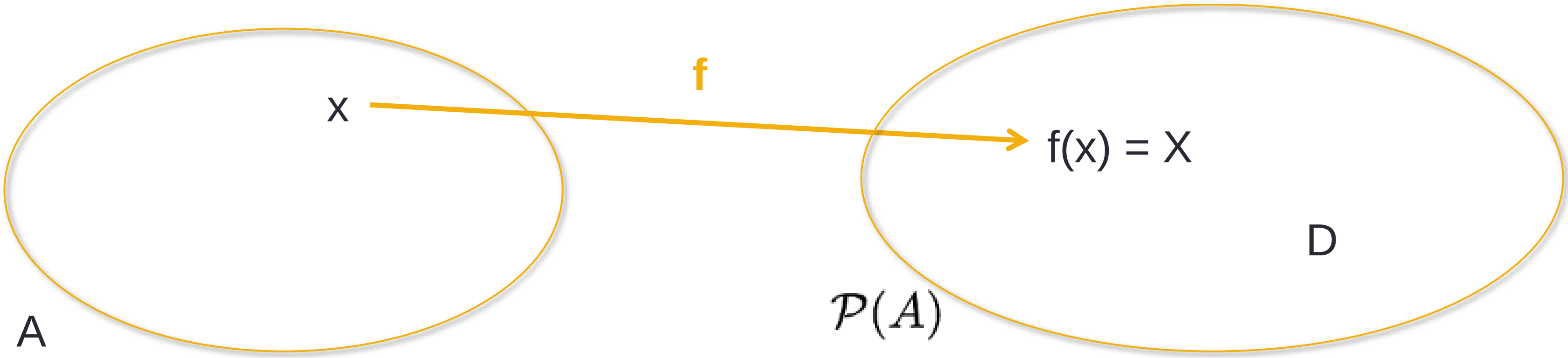


There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \qquad \text{iff} \qquad a \notin f(a)$$



There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \qquad \text{iff} \qquad a \notin f(a)$$

Define d to be the pre-image of D in A under f f(d) = D

Is d in D?

- If yes, then by definition of D, $d \notin f(d) = D$ **a contradiction!**
- Else, by definition of D, $\neg(d \notin f(d))$ so $d \in f(d) = D$ **a contradiction!**

Cardinality

Rosen p. 172

- Uncountable sets — Infinite but not in bijection with $\mathbf{Z}^+$

Examples: the power set of any countably infinite set

and also …

- the set of **real** numbers — Example 5
- (0,1) — Example 6 (++)
- (0,1] — Example 6 (++)

Exercises 33, 34

Happy Thanksgiving!