Reminder: Discussion sections are meeting tomorrow

CSE 20
DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Define and compute the cardinality of a set.
- Use functions to compare the sizes of sets.
- Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
- Explain the central idea in Cantor's diagonalization argument.

Functions

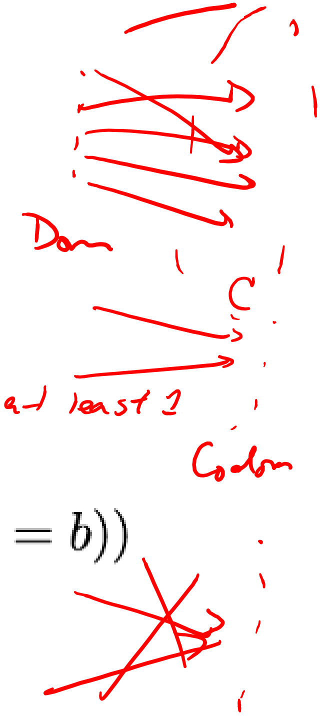
Rosen Sec 2.3; p. 138

Function $\boxed{f: D \rightarrow C}$ means domain D, codomain C, plus rule

Well-defined $\forall a (a \in D \rightarrow \exists! b \, (b \in C \wedge f(a) = b))$

Onto $\forall b (b \in C \rightarrow \exists a \, (a \in D \wedge f(a) = b))$

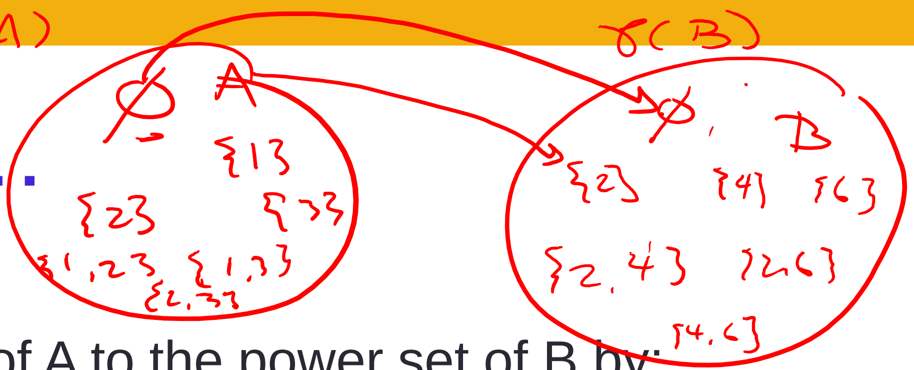
One-to-one $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$



$f(\emptyset) = \emptyset \cap B = \emptyset$

$f(A) = A \cap B = \{1,2,3\} \cap \{2,4,6\} = \{2\}$

Proving a function is …



Let A = {1,2,3} and B = {2,4,6}.

Define a function from the power set of A to the power set of B by:

$$f : \underbrace{\mathcal{P}(A)}_{\text{Dom}} \to \underbrace{\mathcal{P}(B)}_{\text{Codom}}$$

rule $f(X) = X \cap B$ intersection

element of dom

Well-defined? WTS $\forall X\,(X \in \mathcal{P}(A) \longrightarrow X \cap B \in \mathcal{P}(B))$

i.e. $\forall X\,(X \subseteq A \longrightarrow X \cap B \subseteq B)$

Onto?

One-to-one?

A. Onto & 1-1

B. onto, not 1-1

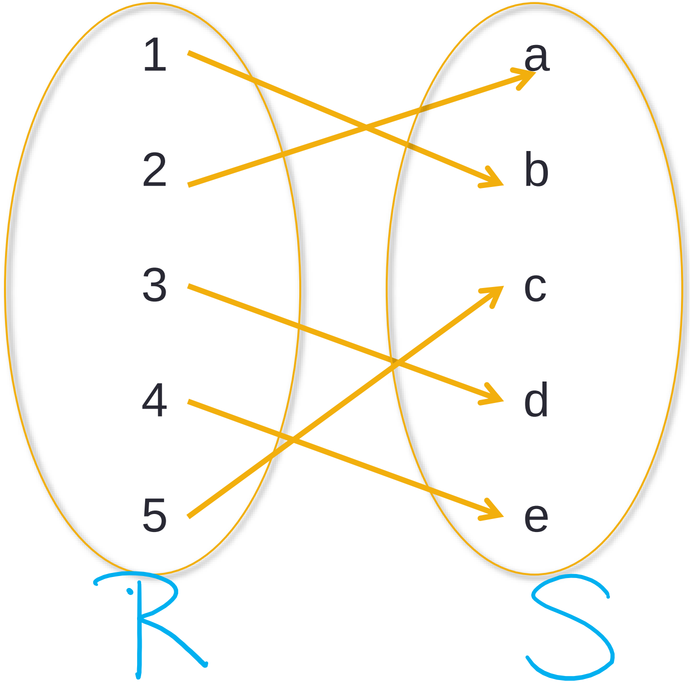
C. not onto, 1-1

D. not onto, not 1-1

$f: R \to S$ $g: S \to R$

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

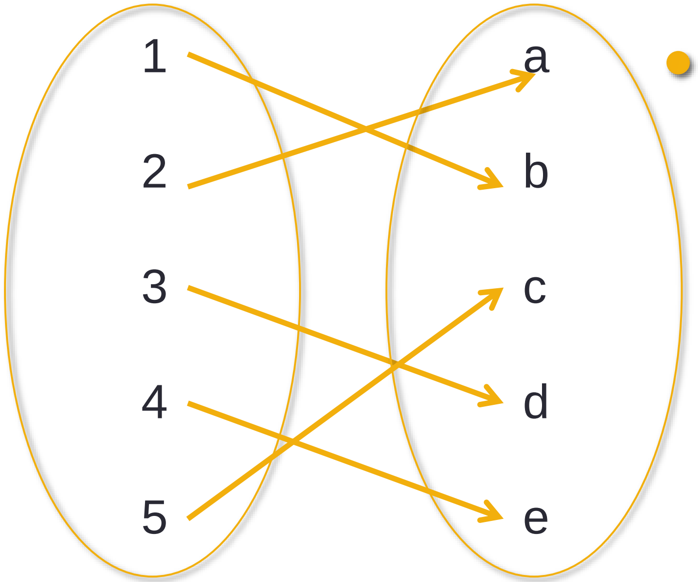
invertible

The **inverse** of a function f: A→B is the function g: B→A such that

$$\forall b(b \in B \to (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144



Fact: for finite sets A and B, there is a bijection between them if and only if |A| =|B|.

Beyond finite sets

Rosen Section 2.5

For all sets, we **define**

$|A| = |B|$ if and only if there is a bijection between them.

Which of the following is true?

A. $|\mathbf{Z}| = |\mathbf{N}|$
B. $|\mathbf{N}| = |\mathbf{Z}^+|$
C. $|\mathbf{Z}| = |\{0,1\}^*|$
D. All of the above.
E. None of the above.

$\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z}$

But $|\mathbb{Z}^+| = |\mathbb{N}| = |\mathbb{Z}|$

Note if have $f: \{0,1\}^* \to \mathbb{Z}^+$ can use $g: \mathbb{Z}^+ \to \mathbb{Z}$

Cardinality

Rosen Defn 3 p. 171

- Finite sets $|A| = n$ for some nonnegative int n
- Countably infinite sets $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

"Smallest" infinite set

Sizes and subsets

*Rosen Theorem 2, p 174 *More on HW**

For all sets A, B we say

$|A| \leq |B|$ if there is a one-to-one function from A to B.

$|A| \geq |B|$ if there is an onto function from A to B.

Cantor-Schroder-Bernstein Theorem: $|A| = |B|$ iff $|A| \leq |B|$ and $|A| \geq |B|$

counterex: choose $A=B$

counterex $A = \mathbb{Z}^+ \subsetneq \mathbb{N} = B$

counterex $A = \{1\} \subsetneq \{1,2\} = B$

Which of the following is true?

A. If A is a subset of B then $|A| \leq |B|$
B. If A is a proper subset of B then $|A| \neq |B|$
C. If A is a subset of B then $|A| = |B|$
D. None of the above.
E. I don't know

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0 \}$

Beyond finite sets

Rosen Section 2.5

For all sets, we say

|A| = |B| if and only if there is a bijection between them.

Which of the following is true?

A. $|\mathbf{Q}| = |\mathbf{Q}^+|$
B. $|\mathbf{Q}^+| = |\mathbf{N} \times \mathbf{N}|$
C. $|\mathbf{N}| = |\mathbf{Q}|$
D. All of the above.
E. None of the above.

Cardinality

Rosen Defn 3 p. 171

- Finite sets — $|A| = n$ for some nonnegative int n
- Countably infinite sets — $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)
- Uncountable sets — Infinite but not in bijection with $\mathbf{Z}^+$

Cardinality

Rosen Defn 3 p. 171

- Finite sets

$|A| = n$ for some nonnegative int n

Which of the following sets is **not** finite?

A. $\emptyset$

B. $[0, 1]$

C. $\{x \in \mathbb{Z} | x^2 = 1\}$

D. $\mathcal{P}(\{1, 2, 3\})$

E. None of the above (they're all finite)

Cardinality

Rosen p. 172

- Countable sets   A is finite or $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$ $\mathcal{P}(\{1,2,3\})$ $\mathbb{Z}^+$

and also …

- the set of **odd positive** integers   Example 1
- the set of **all integers**   Example 3
- the set of **positive rationals**   Example 4
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** {0,1}*

Cardinality

Rosen p. 172

- Countable sets A is finite or $|A| = |Z^+|$ (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$

and also …

- the set of **odd positive** integers
- the set of **all integers**
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Proof strategies?

- List out all and only set elements (with or without duplication)
- Give a one-to-one function from A to (a subset of) a set known to be countable

Proving countability

Which of the following is **not** true?

A. If A and B are both countable then $A \cup B$ is countable.
B. If A and B are both countable then $A \cap B$ is countable.
C. If A and B are both countable then $A \times B$ is countable.
D. If A is countable then P(A) is countable.
E. None of the above

There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

An example to see what is necessary. Consider A = {a,b,c}.

What would we need to prove that |A| = |P(A)|?

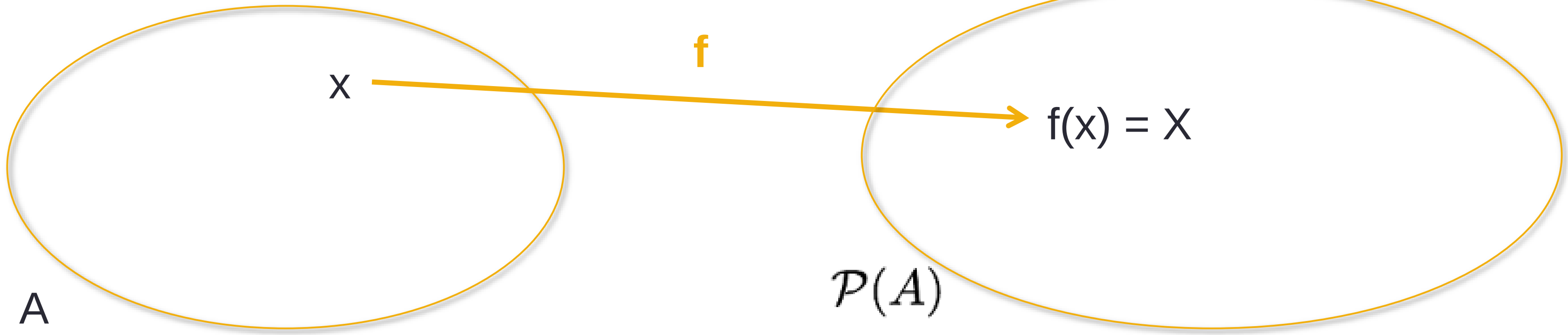
There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

Proof: (Proof by contradiction)

Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \to \mathcal{P}(A)$.

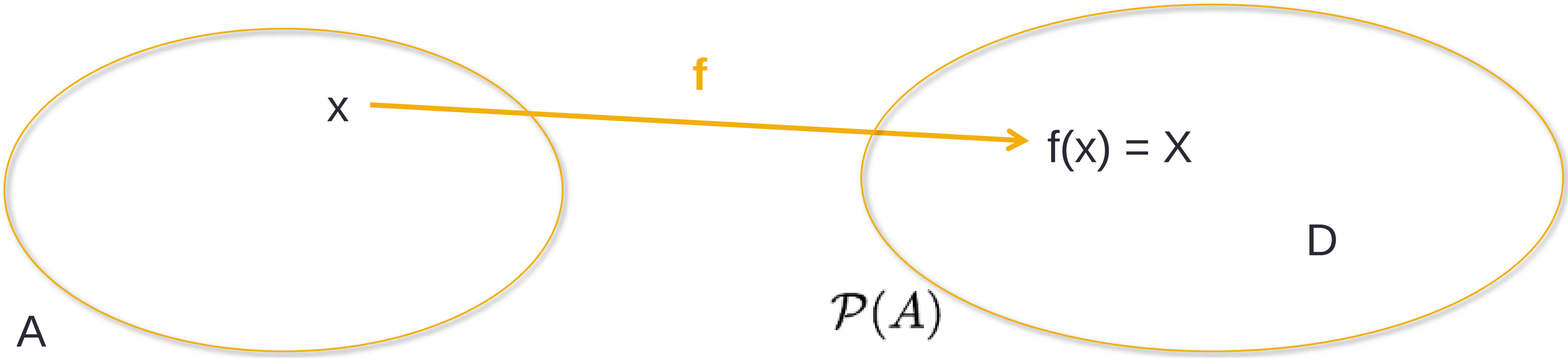


There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \quad \text{iff} \quad a \notin f(a)$$



There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \qquad \text{iff} \qquad a \notin f(a)$$

Define d to be the pre-image of D in A under f f(d) = D

Is d in D?

- If yes, then by definition of D, $d \notin f(d) = D$ **a contradiction!**
- Else, by definition of D, $\neg(d \notin f(d))$ so $d \in f(d) = D$ **a contradiction!**

Cardinality

Rosen p. 172

- Uncountable sets Infinite but not in bijection with $\mathbf{Z}^+$

Examples: the power set of any countably infinite set

and also …

- the set of **real** numbers Example 5
- (0,1) Example 6 (++)
- (0,1] Example 6 (++)

Exercises 33, 34

Happy Thanksgiving!