Today's learning goals

• Define and compute the cardinality of a set.
• Use functions to compare the sizes of sets.
• Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
• Explain the central idea in Cantor's diagonalization argument.
Functions

Function $f: D \rightarrow C$ means domain $D$, codomain $C$, plus rule

Well-defined $\quad \forall a (a \in D \rightarrow \exists! b (b \in C \land f(a) = b))$

Onto $\quad \forall b (b \in C \rightarrow \exists a (a \in D \land f(a) = b))$

One-to-one $\quad \forall a \forall b ((a \in D \land b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$
Proving a function is …

Let \( A = \{1,2,3\} \) and \( B = \{2,4,6\} \).

Define a function from the power set of \( A \) to the power set of \( B \) by:

\[
 f : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \\
 f(X) = X \cap B
\]

Well-defined?
Onto?
One-to-one?
One-to-one + onto

The inverse of a function $f: A \rightarrow B$ is the function $g: B \rightarrow A$ such that

$$\forall b (b \in B \rightarrow (g(b) = a \iff f(a) = b))$$
Fact: for finite sets $A$ and $B$, there is a bijection between them if and only if $|A| = |B|$. 

Rosen p. 144
Beyond finite sets

For all sets, we **define**

$$|A| = |B|$$ if and only if there is a bijection between them.

Which of the following is true?

A. $|\mathbb{Z}| = |\mathbb{N}|$
B. $|\mathbb{N}| = |\mathbb{Z}^+|$
C. $|\mathbb{Z}| = |\{0,1\}^*|$
D. All of the above.
E. None of the above.
Cardinality

- Finite sets
  \[ |A| = n \text{ for some nonnegative int } n \]

- Countably infinite sets
  \[ |A| = |\mathbb{Z}^+| \text{ (informally, can be listed out)} \]

"Smallest" infinite set
Sizes and subsets

For all sets A, B we say

\[ |A| \leq |B| \] if there is a one-to-one function from A to B.

\[ |A| \geq |B| \] if there is an onto function from A to B.

**Cantor-Schroder-Bernstein Theorem:** \[ |A| = |B| \] iff \[ |A| \leq |B| \] and \[ |A| \geq |B| \]

Which of the following is true?

A. If A is a subset of B then \[ |A| \leq |B| \]
B. If A is a subset of B then \[ |A| \neq |B| \]
C. If A is a subset of B then \[ |A| = |B| \]
D. None of the above.
E. I don't know
Beyond finite sets

For all sets, we say $|A| = |B|$ if and only if there is a bijection between them.

Which of the following is true?
A. $|\mathbb{Q}| = |\mathbb{Q}^+|$
B. $|\mathbb{Q}^+| = |\mathbb{N} \times \mathbb{N}|$
C. $|\mathbb{N}| = |\mathbb{Q}|$
D. All of the above.
E. None of the above.
Cardinality

- Finite sets  \[ |A| = n \text{ for some nonnegative int } n \]
- Countably infinite sets  \[ |A| = |\mathbb{Z}^+| \text{ (informally, can be listed out)} \]
- Uncountable sets  \[ \text{Infinite but not in bijection with } \mathbb{Z}^+ \]
Cardinality

- Finite sets

|A| = n for some nonnegative int n

Which of the following sets is **not** finite?

A. Ø
B. [0, 1]
C. \( \{ x \in \mathbb{Z} | x^2 = 1 \} \)
D. \( \mathcal{P} \{1, 2, 3\} \)
E. None of the above (they're all finite)
Cardinality

- Countable sets  \( A \) is finite or \(|A| = |\mathbb{Z}^+|\) (informally, can be listed out)

Examples:
- \( \emptyset \)
- \( \{x \in \mathbb{Z} | x^2 = 1\} \)
- \( \mathcal{P}(\{1, 2, 3\}) \)
- \( \mathbb{Z}^+ \)

and also ...

- the set of **odd positive** integers  Example 1
- the set of **all** integers  Example 3
- the set of **positive rationals**  Example 4
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** \( \{0,1\}^* \)
Cardinality

- Countable sets  \( A \) is finite or \( |A| = |\mathbb{Z}^+| \) (informally, can be listed out)

Examples:  \( \emptyset \), \( \{ x \in \mathbb{Z} | x^2 = 1 \} \)

and also …
- the set of **odd positive** integers
- the set of **all integers**
- the set of **positive rationals**
- the set of **negative rationals**
- the set of **rationals**
- the set of **nonnegative integers**
- the set of **all bit strings** \( \{0,1\}^* \)

Proof strategies?

- List out all and only set elements (with or without duplication)
- Give a one-to-one function from \( A \) to (a subset of) a set known to be countable
Proving countability

Which of the following is not true?

A. If A and B are both countable then $A \cup B$ is countable.
B. If A and B are both countable then $A \cap B$ is countable.
C. If A and B are both countable then $A \times B$ is countable.
D. If A is countable then $P(A)$ is countable.
E. None of the above
There is an uncountable set!  

Rosen example 5, page 173-174

Cantor's diagonalization argument

Theorem: For every set $A$, $|A| \neq |\mathcal{P}(A)|$
There is an uncountable set! *Rosen example 5, page 173-174*

**Cantor's diagonalization argument**

Theorem: For every set $A$, $|A| \neq |\mathcal{P}(A)|$

An example to see what is necessary. Consider $A = \{a,b,c\}$. What would we need to prove that $|A| = |\mathcal{P}(A)|$?
There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

**Theorem:** For every set $A$, $|A| \neq |\mathcal{P}(A)|$

**Proof:** (Proof by contradiction)

Assume *towards a contradiction* that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \rightarrow \mathcal{P}(A)$. 

[Diagram: A set $A$ is mapped to a set $\mathcal{P}(A)$ via a function $f$, where $f(x) = X$.]
There is an uncountable set!  *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset $D$ of $A$ defined by, for each $a$ in $A$:

$$a \in D \iff a \notin f(a)$$

![Diagram showing the concept of Cantor's diagonalization argument](image-url)
There is an uncountable set!  

Cantor's diagonalization argument

Consider the subset $D$ of $A$ defined by, for each $a$ in $A$:

$$a \in D \text{ iff } a \notin f(a)$$

Define $d$ to be the pre-image of $D$ in $A$ under $f$:

$$f(d) = D$$

Is $d$ in $D$?

- If yes, then by definition of $D$, $d \notin f(d) = D$  
  a contradiction!
- Else, by definition of $D$, $\neg (d \notin f(d))$ so $d \in f(d) = D$  
  a contradiction!
Cardinality

• Uncountable sets

Examples: the power set of any countably infinite set and also …
- the set of real numbers
- (0,1)
- (0,1]

Exercises 33, 34
Happy Thanksgiving!