CSE 20
DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Explain the steps in a proof by (strong) mathematical induction
- Use (strong) mathematical induction to prove
  - correctness of identities and inequalities
  - properties of algorithms
  - properties of geometric constructions
- Represent functions in multiple ways
- Use functions to define sequences: arithmetic progressions, geometric progressions
- Use and prove properties of recursively defined functions and recurrence relations (using induction)
- Evaluate which variant of induction is appropriate in a given situation, and apply it accordingly
- Find mistakes in purported proofs using induction and explain why they are wrong.

Thm: $\forall k\, P(k)$ where Domain $\mathbb{Z}^+$

MI

Base Case

Let k be arb

Assume $P(k)$

WTS $P(k+1)$

Thm: $\forall k\, P(k)$ where domain S

Str. Ind

Rec Step Let x be element

Assume $P(x)$ WTS still true

Recursive definitions

Rosen Sec 5.3 page 349

The set of **bit strings** $\{0,1\}^*$ is defined recursively by

Basis step: $\lambda \in \{0,1\}^*$ where λ is the empty string.

Recursive step: If $w \in \{0,1\}^*$, then $w0 \in \{0,1\}^*$ and $w1 \in \{0,1\}^*$

How many strings are formed using the basis step and at most k applications of the recursive step (k>=1)?

A. 1
B. K
C. k^2+k+1
D. 2^k+1
E. None of the above

Recursive definitions

Rosen Sec 5.3 page 349

The set of **bit strings** $\{0,1\}^*$ is defined recursively by

Basis step: $\lambda \in \{0,1\}^*$ where λ is the empty string.

Recursive step: If $w \in \{0,1\}^*$, then $w0 \in \{0,1\}^*$ and $w1 \in \{0,1\}^*$.

Where does the notation {0,1}* come from? (cf. HW 6 Q2)

Kleene star applied to set A gives a set, written A*, defined as:

Basis step: $\lambda \in A^*$

Recursive step: If u is in A* and v is in A, then $uv \in A^*$

Recursive definition: rooted trees *Rosen p. 351*

The set of **rooted trees** is defined recursively:

Basis step : A single vertex r is a rooted tree.

Recursive step : Suppose that $T_1 ,\ldots, T_n$ are disjoint rooted trees with roots $r_1, \ldots, r_n$ (respectively). Then there is a rooted tree consisting of a (new) root together with edges connecting this root to each of the vertices $r_1, \ldots, r_n$.

Recursive definition: eb trees *Rosen p. 352*

Basis step : ·

Recursive step

1. *What can $T_1 \ldots T_n$ be?*

2. *What can $T_1 \ldots T_n$ be?*

Recursive definition: rooted trees *Rosen p. 351*

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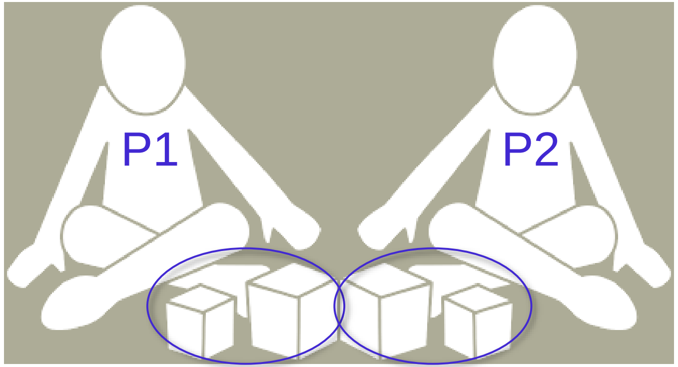
How many rooted trees are formed using the basis step and at most k applications of the recursive step (k>=1)?

A. 1
B. k
C. k^2
D. 2^k
E. None of the above

And now for something completely different …

(except not really)

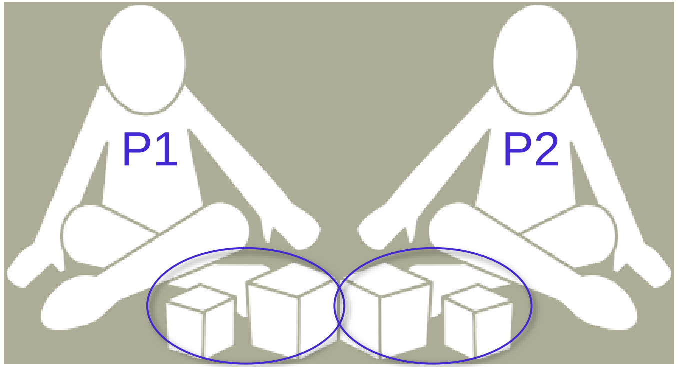
Nim



Two players start with two (equal) piles of jellybeans in front of them. They take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them.

The player who removes the **last jellybean wins** the game.

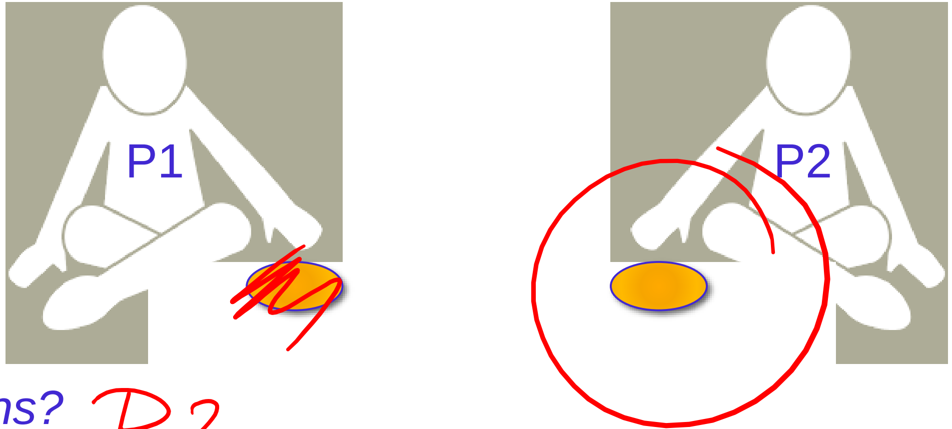
Nim



Two players start with two (equal) piles of jellybeans in front of them.
They take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them.
The player who removes the **last jellybean wins** the game.
Who has a strategy to guarantee a win?

Nim

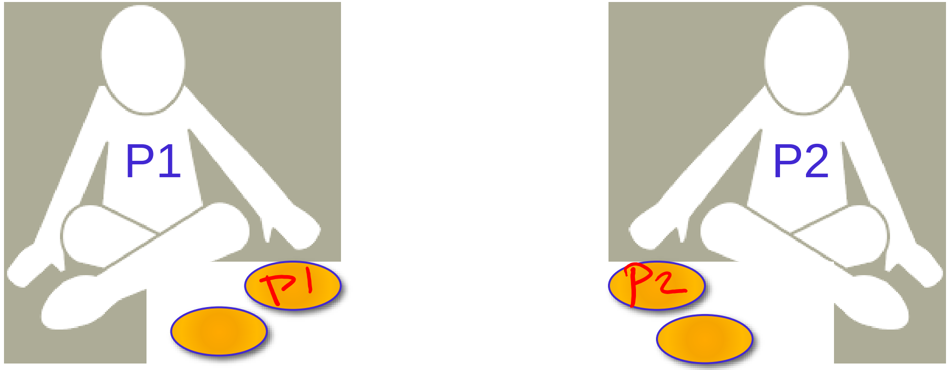
The piles start out with equal #s. Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. The player who removes the **last jellybean wins** the game.



n=1 *Who wins?* P2

Nim

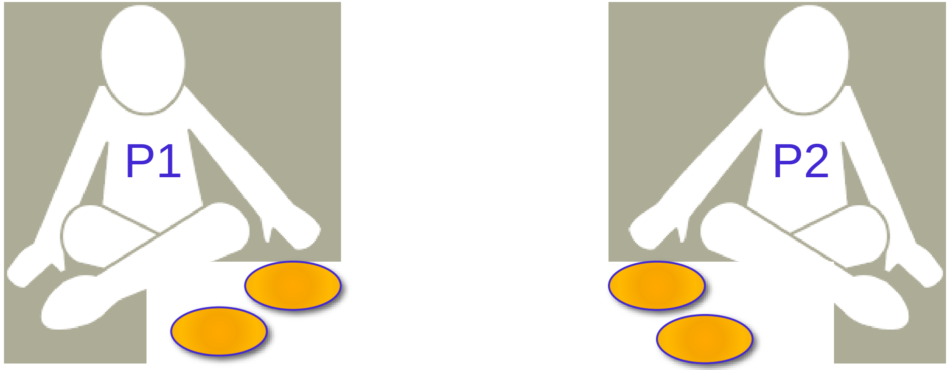
Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.



n=2 *Who wins?* P2

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.



Idea: 2^{nd} player takes the same amount 1^{st} player took but from opposite pile.

*…Game reduces to **same setup** but with fewer jellybeans.*

Strong induction

Rosen p. 334

To show that some statement P(n) is true about all positive integers n,

1. Verify that P(1) is true.
2. Let k be an arbitrary positive integer. Show that

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$$

is true.

Strong induction hypothesis

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.

Theorem: the second player can always guarantee a win.

Proof: By Strong Mathematical Induction, on # jellybeans in each pile.

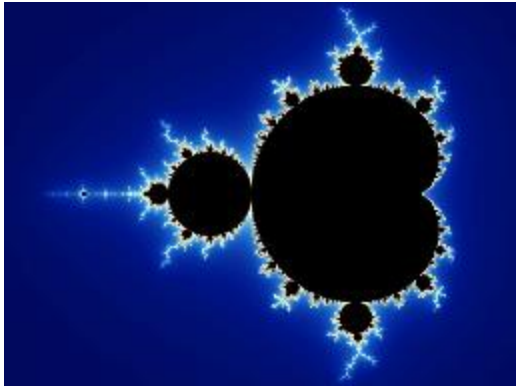
1. Basis step WTS if piles each have 1, then 2^{nd} player can win.
2. Induction step Let k be a positive integer.

As the Strong Induction Hypothesis Assume that 2^{nd} player can win whenever there are j jellybeans in each pile, for each j between 1 and k (inclusive).

WTS 2^{nd} player has winning strategy when start with k+1 jellybeans in each pile.

How does this connect?

Rosen Sec 5.3



n!	$0! = 1$,	$n! = n\,(n-1)!$	for $n > 0$.
2^n	$2^0 = 1$,	$2^n = 2\,(2^{n-1})$	for $n > 0$.

2, -8, 32, -128, 512, …

	$a_0 = 2$,	$a_n = -4\,a_{n-1}$	for $n > 0$.

Recursive definitions can refer to **much** earlier terms / values

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1, \; f_1 = 1, \; f_n = f_{n-1} + f_{n-2}$$

$f_0 = 1$

$f_1 = 1$

$f_2 = f_1 + f_0 = 2$

$f_3 = f_2 + f_1 = 3$

What are some sample values?

How quickly do these value grow?

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

Theorem: For each integer n>=2, $f_n >= 1.5^{n-2}$

Proof: By Strong Mathematical Induction, on n>=2.

1. Basis step WTS $f_2 >= 1.5^{2-2}$.
2. Induction step Let k be an integer, k>=2. Assume as the **strong induction hypothesis** that the inequality is true about f_j for each integer j, 2<=j<=k**.** WTS statement is true about f_{k+1}.

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1, \; f_1 = 1, \; f_n = f_{n-1} + f_{n-2}$$

1. Basis step WTS $f_2 >= 1.5^{2-2}$.

LHS = f_2 = 1 + 1 = 2.

RHS = $1.5^{2-2} = 1.5^0 = 1$.

Since 2>1, LHS>RHS so, in particular, LHS>= RHS ☺

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

Induction step Let k be an integer with k>=2.

Assume as the **strong induction hypothesis** that

$$f_j >= 1.5^{j-2}$$

for each integer j with 2<=j<=k.

WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$

By definition of Fibonacci numbers, since k+1 > 1, $f_{k+1} = f_k + f_{k-1}$.

Therefore, LHS = $f_{k+1} = f_k + f_{k-1}$.

Idea: apply strong induction hypothesis to k and k-1. Can we do it?

Fibonacci numbers

Rosen p. 158, 347

…

Case 1: k=2 and WTS that $\mathbf{f_3 >= 1.5^{(3)-2}}$

Case 2: k>2 and WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$ and strong IH applies to k, k-1 (because both k, k-1 are greater than or equal to 2 and less than k+1).

So let's prove each of these cases in turn:

Case 1: k=2 and WTS that $\mathbf{f_3 >= 1.5^{(3)-2}}$

By definition of Fibonacci numbers, LHS = $f_3=f_2+f_1=2+1=3$.

By algebra, RHS = $1.5^{3-2}=1.5$ Since 3>1.5, LHS>RHS ☺

Fibonacci numbers

Rosen p. 158, 347

…

Case 2: $k>2$ and WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$ and strong IH applies to k, k-1 (because both k, k-1 are greater than or equal to 2 and less than k+1).

LHS = $f_{k+1} = f_k + f_{k-1} >= 1.5^{k-2} + 1.5^{(k-1)-2} = 1.5^{k-3}(1.5+1) = 1.5^{k-3}(2.5)$

$> 1.5^{k-3}(2.25) = 1.5^{k-3}1.5^2 = 1.5^{k-1} = 1.5^{(k+1)-2}$ = RHS.

Def of Fibonacci numbers

Strong induction hypothesis

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

Theorem: For each integer n>=2, $f_n >= 1.5^{n-2}$

Proof: By Strong Mathematical Induction, on n>=2.

1. Basis step WTS $f_2 >= 1.5^{2-2}$ and $f_3 >= 1.5^{3-2}$
2. Induction step Let k be an integer, k>=3. Assume as the **strong induction hypothesis** that the inequality is true about f_j and f_{j-1} for each integer j, 2<=j<=k. WTS statement is true about f_{k+1}.

Flavors of induction

- Mathematical induction

  $\forall k \, (P(k) \rightarrow P(k+1))$

  Domain : $\{x \in \mathbb{Z} \mid x \geq b\}$

- Structural induction

  Domain any rec def set

- Strong induction

  refer back further than "1" away

  Domain $\{x \in \mathbb{Z} \mid x \geq b\}$

Looking back

- We now have all the tools we need to rigorously prove
  - Correctness of **greedy change-making algorithm** with quarters, dimes, nickels, and pennies *Proof by contradiction, Rosen p. 199*
  - The **division algorithm** is correct *Strong induction, Rosen p. 341*
  - **Russian peasant multiplication** is correct *Induction*
  - Largest **n-bit binary** number is 2^n-1 *Induction, Rosen p. 318*
  - Correctness of **base b conversion** (Algorithm 1 of 4.2), *Strong induction*
  - Size of the **power set** of a finite set with n elements is 2^n *Induction, Rosen p. 323*
  - Any int greater than 1 can be written as **product of primes** *Strong induction, Rosen p. 323*
  - There are infinitely many **primes** *Proof by contradiction, Rosen p. 260*
  - **Sum** of geometric progressions $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$ when r≠1, *Induction, Rosen p. 318*

Cautionary tales

- The **basis step** is absolutely necessary … and might need more than one!
- Make sure to stay in the **domain**.

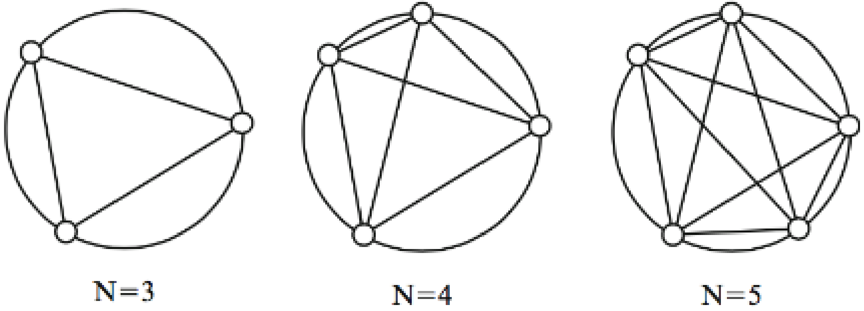
Recommended practice

Section 5.1 #49, 50, 51

Section 5.2 #32

- A few **examples** do not guarantee a pattern:

cake cutting conundrum. Join all pairs of points among N marked on circumference of cake.



Reminders

- Discussion tomorrow
- HW6 available