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CSE 20 DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Explain the steps in a proof by mathematical and/or structural induction
- Use a recursive definition of a set to determine the elements of the set
- Write a recursive definition of a given set
- Use structural induction to prove properties of a recursively defined set
- Define functions from **N** to **R** using recursive definitions

Mathematical induction

Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k,

1. Show that it's true about 0 i.e. P(0)
2. Show $\forall k \; (\; P(k) \rightarrow P(k+1) \;)$ Hence conclude P(1),…

Mathematical induction*

Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k >= b,

1. Show that it's true about b i.e. P(b)
2. Show $\forall k \; (\; P(k) \to P(k+1) \;)$ Hence conclude P(b+1),…

Sizes of (finite) sets

If S is a set with exactly *n* **distinct** elements, with *n* a nonnegative integer, then S is **finite set** and |S| = n.

$|\{\,\}| = 0$ $\quad |\{1\}| = 1$ $\quad |\{1,1,2,3,8/4\}| = 3$

For A, B finite sets:

The picture can't be displayed.

$|A \cup B| = |A| + |B| - |A \cap B|$

- $|A \times B| = |A| \cdot |B|$
- if A has size n, then the power set of A has size 2^n.

An **infinite** set is a set that is not finite

Recursive definitions, part 1

Rosen Sec 5.3

To specify a **function**,
we need to specify its

1. domain – input
2. codomain – type of outputs
3. assignment / rule – formula, table of values, induction

Rosen p. 138

Recursive definitions, part 1

Rosen Sec 5.3

For functions $f: \mathbf{N} \rightarrow X$

- define by a (closed-form) formula

$f(0)$ $f(1)$ $f(2)$ $f(3)$ $f(4)$

1. domain – **N**
2. codomain – X
3. assignment / rule – formula, table of values, induction

Recursive definitions, part 1

Rosen Sec 5.3

For functions f: **N** → X

- define by a (closed-form) formula

Sequences are functions too!

f(0)	f(1)	f(2)	f(3)	f(4)	…
a_0	a_1	a_2	a_3	a_4	…

1. domain – **N**
2. codomain – X
3. assignment / rule – formula, table of values, induction

BA

Recursive definitions, part 1

Rosen Sec 5.3

For functions f: **N** → X

- define by a (closed-form) formula or …

Basis step: Specify the value of the function at 0

Recursive step: Give a rule for finding its value at an integer from its values at smaller integers

Which of the following functions / sequences have recursive definitions?

A. n!

B. 2^n

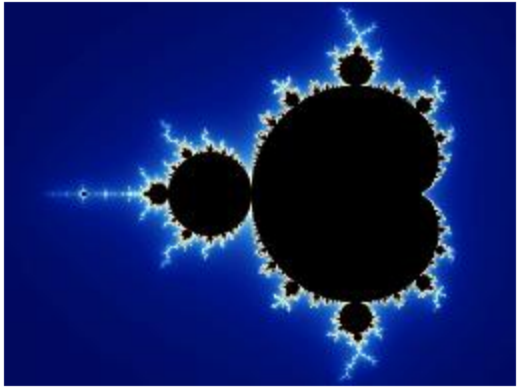
C. 2, -8, 32, -128, 512, …

D. $\sum_{i=1}^{n}(i^2+i)$

E. All of the above.

Recursive definitions, part 2

Rosen Sec 5.3



n!	$0! = 1,$	$n! = n\,(n-1)!$	for $n > 0$.
2^n	$2^0 = 1,$	$2^n = 2\,(2^{n-1})$	for $n > 0$.

2, -8, 32, -128, 512, …

$a_0 = 2,$ $\quad a_n = -4\,a_{n-1}$ $\quad$ for $n > 0$.

terms

$$\sum_{i=1}^{n} g(n) = g(1) + g(2) + \cdots + g(n)$$

new term

$$\boxed{\sum_{i=1}^{0} g(i) = 0}, \quad \sum_{i=1}^{1} g(i) = g(1), \text{ and } \underbrace{\sum_{i=1}^{n} g(i) = g(n) + \sum_{i=1}^{n-1} g(i) \text{ for } n>1.}_{\text{Rec.}}$$

Basis

Extra mathematical induction practice

1 $n^2 < n!$ for all n greater than or equal to …

2 $1+2+3+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Video

3 $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Recursive definitions, part 2

Rosen Sec 5.3

Define a set S by

$\mathbb{N} =$
$\{0, 1, 2, \ldots\}$

{ … } roster

{ x | P(x) } set builder

Recursive definition:

Basis step – Specify initial collection of elements. 0

Recursive step – Provide rules for forming new elements in the set from those already known to be in the set.

If $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$

Example: $S = \{ x \mid x \text{ is an int}, x > 3\}$

Basis Step: $\boxed{4 \in S \quad 17 \in S}$

Rec Step: If $n \in S$ then $n+1 \in S$

Recursive definitions, part 2

Rosen Sec 5.3

Let S be the subset of the set of **integers** defined recursively by

Basis step: $1 \in S$

Recursive step: If $a \in S$, then $a + 2 \in S$.

$S = \{1, 3, 5, \ldots\}$

What's an equivalent description of this set?

A. All positive multiples of 2.

B. All positive even integers.

C. All positive odd integers.

D. All integers.

E. None of the above.

$|\{0, 00\}| = 2$

Recursive definitions, part 2

Rosen Sec 5.3 page 349

The set of **bit strings** {0,1}* is defined recursively by

Basis step: $\lambda \in \{0,1\}^*$ where λ is the empty string.

Recursive step: If $w \in \{0,1\}^*$, then $w0 \in \{0,1\}^*$ and $w1 \in \{0,1\}^*$.

Set of bit strings

$w = \lambda$
$w0 = 0$

$w = \lambda$
$w1 = 1$

$= \{ \lambda, 0, 1, 00, 01, 10, 11, \ldots, 000 \}$

$w = 0$
$w0 = 00$

Also known as
binary sequences

Recursive definitions, part 2 *Rosen Sec 5.3 page 349*

The set of **bit strings** $\{0,1\}^*$ is defined recursively by

Basis step: $\lambda \in \{0,1\}^*$ where λ is the empty string.

Recursive step: If $w \in \{0,1\}^*$, then $w0 \in \{0,1\}^*$ and $w1 \in \{0,1\}^*$.

Which of the following are **not** bit strings?

~~A.~~ λ

B. 1

C. 000

D. (01010)1 = ((   )0)1 = · · · = (λ · · · ·))

E. 10110011100011110000111110000011…

Also known as **binary sequences**

infinite sequence!

Recursive definitions in CS

- Strings – encode integers, everything else
- Data structures – linked lists, trees, graphs

*How do we prove a fact is true about **all** strings?*

Structural induction

Rosen p. 354

To show that some statement P(k) is true about all elements of a recursively defined set S,

1. Show that it's true for each element specified in the **basis step** to be part of S.
2. Show that if it's true for each of the elements used to construct **new elements in recursive step**, then it holds for these new elements.

Structural induction, example

Rosen Sec 5.3

Define the **subset S** of the set of all bit strings, {0,1}*, by

Basis step: $\boxed{\lambda \in S}$ where λ is the empty string.

Recursive step: If $w \in S$, then each of $10w \in S, 01w \in S$

$\{\lambda, \underbrace{10, 01}_{\text{1 appl'n rec. step}}, 1010, 0110, 1001, 0101, \ldots\}$

Structural induction, example

Rosen Sec 5.3

Define the **subset S** of the set of all bit strings, $\{0,1\}^*$, by

Basis step: $\lambda \in S$ where λ is the empty string.

Recursive step: If $w \in S$, then each of $10w \in S, 01w \in S$

Claim: Every element in S has an equal number of 0s and 1s.

Pf by Structural Induction

Basis Step WTS λ has equal # of 0s, 1s

Note λ has 0 0s, 0 1s, so equal #

Structural induction, example

Rosen Sec 5.3

Define the **subset S** of the set of all bit strings, $\{0,1\}^*$, by

Basis step: $\lambda \in S$ where λ is the empty string.

Recursive step: If $w \in S$, then each of $10w \in S, 01w \in S$

Claim: Every element in S has an equal number of 0s and 1s.

Proof: *Basis step – WTS that empty string has equal # of 0s and 1s*

Recursive step – Let w be an arbitrary element of S.

Assume, as the ***IH*** *that w has equal # of 0s and 1s.*

WTS that 10w, 01w each have equal # of 0s, 1s.

ex.

Recursive definitions, part 3

Rosen Sec 5.3

Define a function f: **S** → X by

- specifying f(a) when a is in the basis collection of elements of S
- giving a formula for computing f(x) when x is built from other elements a_1, a_2 etc. based on values of $f(a_1)$, $f(a_2)$, etc.

1. domain – **S,** a recursively defined set
2. codomain – X
3. assignment / rule – recursively defined

Recursive definitions, part 3

Rosen Sec 5.3

Define a function f: **S** → X by

- specifying f(a) when a is in the basis collection of elements of S
- giving a formula for computing f(x) when x is built from other elements a_1, a_2 etc. based on values of $f(a_1)$, $f(a_2)$, etc.

Example: how do you define the length function l: {0,1}* → N ?

Basis Rec.

λ w

w0

w1

Reminders

- Office hours may be shifted – check Google calendar
- HW5 due Saturday night

Next class: more applications + variants of induction