Feedback
Survey
out

CSE 20
DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/

Today's learning goals

- Explain the steps in a proof by mathematical induction
- Use mathematical induction to prove
  - correctness of identities and inequalities
  - properties of algorithms
  - properties of geometric constructions

Induction: a road map

- Today
  - What is a proof by induction?
  - Examples: inequalities, algorithms, constructions
- Thursday
  - Strong induction
  - Recursive definitions: functions, sets, sigma notation
  - More examples / proofs: identities, constructions

Proof strategies so far

fixed unknown

Theorem: $\forall x P(x)$ over a given domain.

Strategy (1): Let x be arbitrary element of the domain. WTS P(x) is true.

Strategy (2) if domain finite: Enumerate all x in domain. WTS P(x) is true.

Strategy (3) Proof by contradiction: Assume there is an x with P(x) false. WTS badness!

Theorem: $\exists x P(x)$ over a given domain.

Strategy (1): Define x = …. (some specific element in domain) WTS P(x) is true.

Strategy (2) Proof by contradiction: Assume that for all x, P(x) is false. WTS badness!

Theorem: $P \to Q$ over a given domain.

Strategy (1): Toward direct proof, assume P and WTS Q.

Strategy (2): Toward proof by contrapositive, assume Q is false and WTS P is also false.

Strategy (3) Proof by contradiction: Assume both P is true and Q is false. WTS badness!

$\forall k\ P(k)$

$\boxed{0}, 1, 2, 3, \ldots\ldots n$

+1

A new proof strategy

Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k,

1. Show that it's true about 0 — i.e. P(0)
2. Show P(0) → P(1) — Hence conclude P(1)
3. Show P(1) → P(2) — Hence conclude P(2)
4. Show P(2) → P(3) — Hence conclude P(3)
5. …..

A new proof strategy

Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k,

Both important for induction to be applicable

1. Show that it's true about 0
2. Show P(0) → P(1)
3. Show P(1) → P(2)
4. Show P(2) → P(3)
5. …..

i.e

Hence conclude P(1)

Hence conclude P(2)

Hence conclude P(3)

Mathematical induction

Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k,

1. Show that it's true about 0   i.e. P(0)
2. Show $\boxed{\forall k \ (P(k) \rightarrow P(k+1))}$ Hence conclude P(1),…

universal conditional

An inequality

$$\left(1+\frac{1}{2}\right)^{n} \geq 1+\frac{n}{2}$$

Theorem: This inequality is true for all nonnegative integers

$\forall k\ P(k)$

Proof: by Mathematical Induction.

substitut k for all instances of n

What's P(k)?

A. $\left(1+\frac{1}{2}\right)^{\cancel{n}\,k} \geq 1+\frac{\cancel{n}\,k}{2}$

~~B.~~ $\left(1+\frac{1}{2}\right) \geq 1+\frac{n}{2}$

~~C.~~ $\left(1+\frac{1}{2}\right)^{k}$

~~D.~~ $\left(1+\frac{n}{2}\right)^{k}$

E. None of the above.

An inequality $\left(1+\frac{1}{2}\right)^n \geq 1+\frac{n}{2}$

Theorem: This inequality is true for all nonnegative integers

Proof: by Mathematical Induction.

1. Basis step $\boxed{\text{WTS}}$ $\left(1+\frac{1}{2}\right)^0 \geq 1+\frac{0}{2}$

   $P(0)$

2. Inductive step Let k be a nonnegative integer. Assume

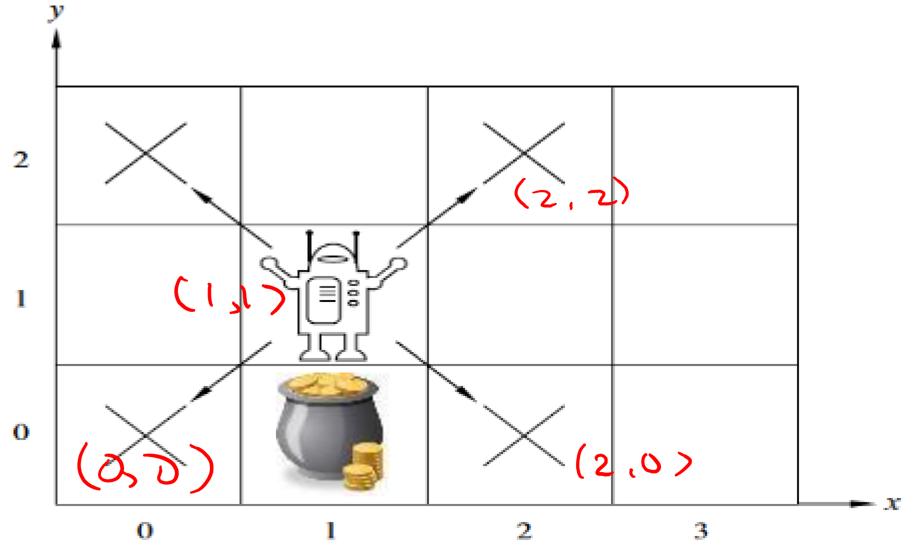
   $$\left(1+\frac{1}{2}\right)^k \geq 1+\frac{k}{2}$$

   $\forall k\,(P(k) \rightarrow P(k+1))$

   $\boxed{\text{WTS}}$ $\left(1+\frac{1}{2}\right)^{k+1} \geq 1+\frac{k+1}{2}$

Robot

Start at origin, moves on infinite 2-dimensional integer grid.
At each step, move to diagonally adjacent grid point.



Can it ever reach (1,0)?

Robot

Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof of Lemma: by mathematical induction on the number of steps.

Rewrite invariant as

$\forall n$ (Posn of robot after n steps has n)

Using proof: Sum of coordinates of (1,0) is 1 so not even!

Proof of Invariant

Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof: by Mathematical Induction.

1. Basis step WTS The sum of the coordinates reachable after 0 steps is even.

$P(0)$

At start, robot is at $(0,0)$.

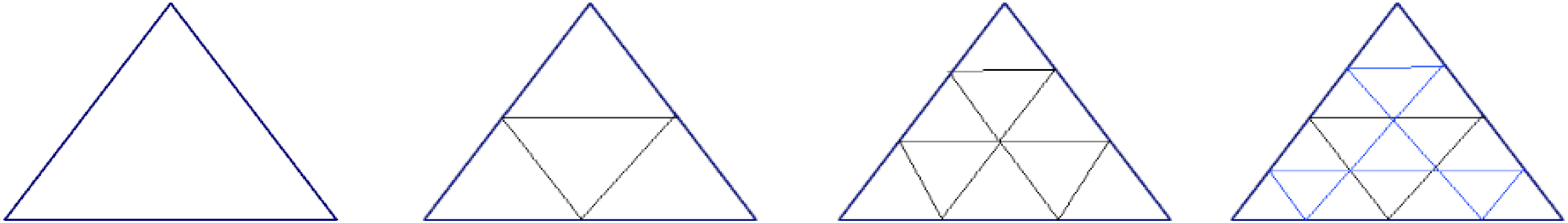
Sum of coords $= 0+0 = 0 = 2 \cdot \underset{\text{int}}{0}$, so even :)

2. Inductive step Let k be a nonnegative integer. Assume the sum of the coordinates of any state reachable after k steps is even.

$\forall k\,(P(k) \to P(k+1))$

WTS the sum of the coordinates of any state reachable after k+1 steps is even.

Triangles



Theorem: If start with an equilateral triangle and divide each side into n equal segments, then connect the division points with all possible line segments parallel to sides of original triangle, then ______ many small triangles will be contained in the original triangle.

Triangles

Theorem: At stage n, n^2 triangles result.

Proof: by Mathematical Induction.

1. Basis step WTS that when side lengths undivided, 1^2 triangles are formed.
2. Inductive step Let k be a nonnegative integer. Assume that when divide sides into k pieces, k^2 triangles result..

WTS when divide sides into k+1 pieces, $(k+1)^2$ triangles result.

Sizes of (finite) sets

If S is a set with exactly *n* **distinct** elements, with *n* a nonnegative integer, then S is **finite set** and |S| = n.

Which of the following sets are finite?
Assume universe is set of real numbers.

A. $\emptyset$ no elements!

~~B.~~ $\mathbb{Q} \cap \{x \mid 0 \leq x \leq 1\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{\sqrt{2}}{\sqrt{2}}, \ldots$ - - -

C. $\mathbb{Z} \cap \{x \mid 0 \leq x \leq 1\} = \{0, 1\}$

~~D.~~ $\mathbb{Z} \cup \{x \mid 0 \leq x \leq 1\} = \{0, 1, 2, \ldots, -2, -3, \ldots, \frac{1}{2}, \frac{1}{\sqrt{2}}$

E. None of the above.

An **infinite** set is a set that is not finite

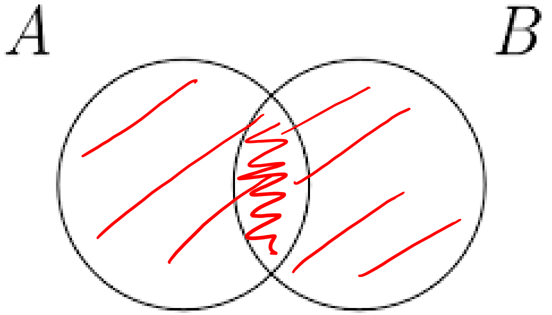
Operations on sets

Rosen Sections 2.1, 2.2

If the sets A, B are finite then

$$|A \cup B| = ?$$

A. |A| + |B| − |A∩B|
B. |A| - |B|
C. |A| |B|
D. $|A|^{|B|}$
E. None of the above.

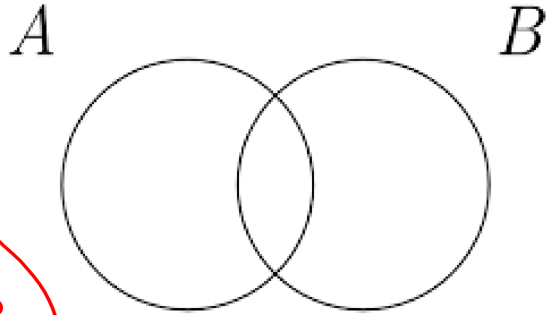


Operations on sets

Rosen Sections 2.1, 2.2

If the sets A, B are finite then

$$|A \cup B| = |A| + |B| - |A \cap B|$$



If A, B are disjoint
($A \cap B = \emptyset$ i.e. $|A \cap B| = 0$)

then $|A \cup B| = |A| + |B|$

Operations on sets

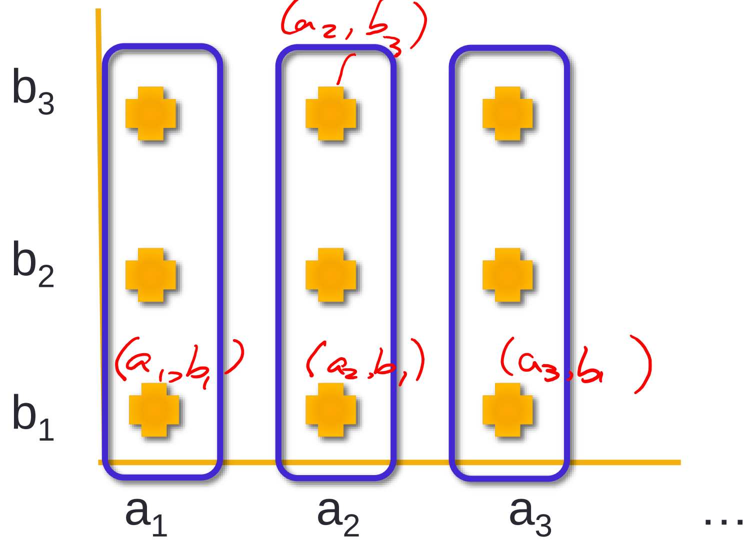
Rosen Sections 2.1, 2.2

$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$

If the sets A, B are finite then

$$\boxed{|A \times B| = |A| \cdot |B|}$$

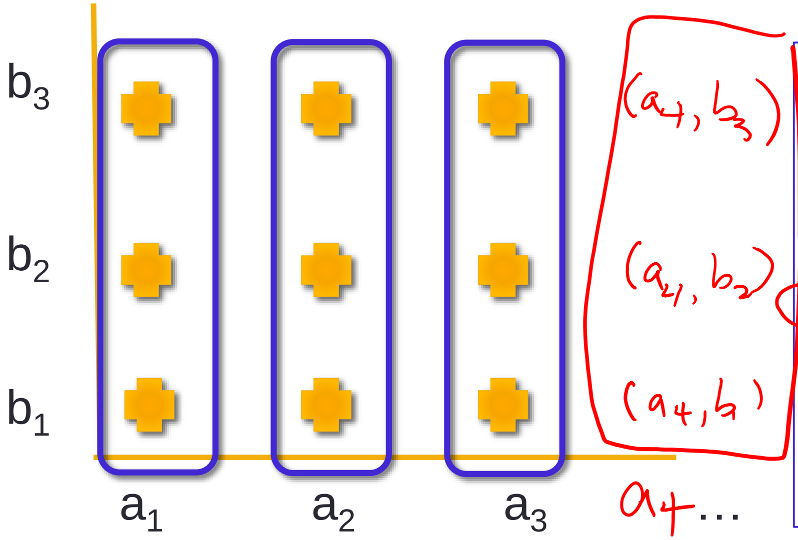
|B| many elements for each of the |A| many elements in A



$\{(a_1, b_1), (a_1, b_2), \ldots\}$

How do we prove this general formula?

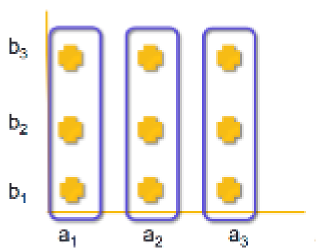
$$|A \times B| = |A| \cdot |B|$$



How much does |AxB| change when we add one new element to A?

A. Add one element.
B. Add |B| many elements.
C. Multiply number of elements by |B|.
D. Raise number of elements to $|B|^{th}$ power.
E. None of the above.

How do we prove this?



Theorem: For any nonnegative integer n, if A is any set of size n, then for any finite set B, |AxB| = |A||B|.

Proof by Mathematical Induction:

n=0 means

1. Basis step WTS if A is empty, then for any finite set B,

   WTS P(0)

   $|AxB| = |A||B| = 0.$ by alg b/c |A|=0

   def of Cart. prod

2. Inductive step Let k be a nonnegative integer. Assume that for any set C of size k and any finite set D, |CxD| = |C||D|.

   WTS for any set A of size k+1 and any finite set B, |AxB| = |A||B|.

Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

disjoint sets!

Let B be any finite set. Then

$$A \times B = (C \times B) \cup \{ (x_{k+1}, y) \mid y \text{ is in } B\}$$

so

$$|A \times B| = |C \times B| + |B| = k\,|B| + |B| = (k+1)\,|B = |A|\,|B|$$

Operations on sets

Rosen Sections 2.1, 2.2

Power set: For a set S, its power set is the set of all subsets of S. $\mathcal{P}(S) = \{A \mid A \subseteq S\}$

If the set S is finite then …

*Does the **size** of the power set of S depend **just** on the size of S?*

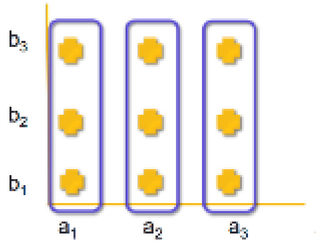
Building up the power set

Set	Power set
{a}	{ emptyset , {a} }
{a,b}	{ emptyset, {a}, {b}, {a,b} }
{a,b,c}	{ emptyset, {a}, {b}, {a,b} {c}, {a,c}, {b,c}, {a,b,c} }

Observe: the size of the power set increases by a factor of 2 for every new element of the set.

$$|\mathcal{P}(A)| = 2^{|A|}$$

How do we prove this?



Theorem: For any nonnegative integer n, if A is any set of size n, then the power set of A has size 2^n.

Proof by Mathematical Induction:

1. Basis step WTS if A is empty, then its power set has size 2^0.
2. Inductive step Let k be a nonnegative integer. Assume that the power set of any set C of size k has size 2^k.

   WTS for any set A of size k+1, its power set will have size 2^{k+1}.

Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

What are the subsets of A?

Reminders

- Discussion section tomorrow
- Review Quiz for this week due tomorrow night
- Office hours may be shifted – check Google calendar
- HW5 due Saturday night

Next class: recursive definitions and variants of induction